Lecture 9

Poisson distribution

Wombats

Poisson process

Exponential distribution

Poisson approximation to binomial

$X \sim \text{Bin}(n, p)$ with n large, p small, and np reasonable, $\Rightarrow$

$$P\{X = k\} \;\approx\; \frac{e^{-\mu}\mu^k}{k!},$$

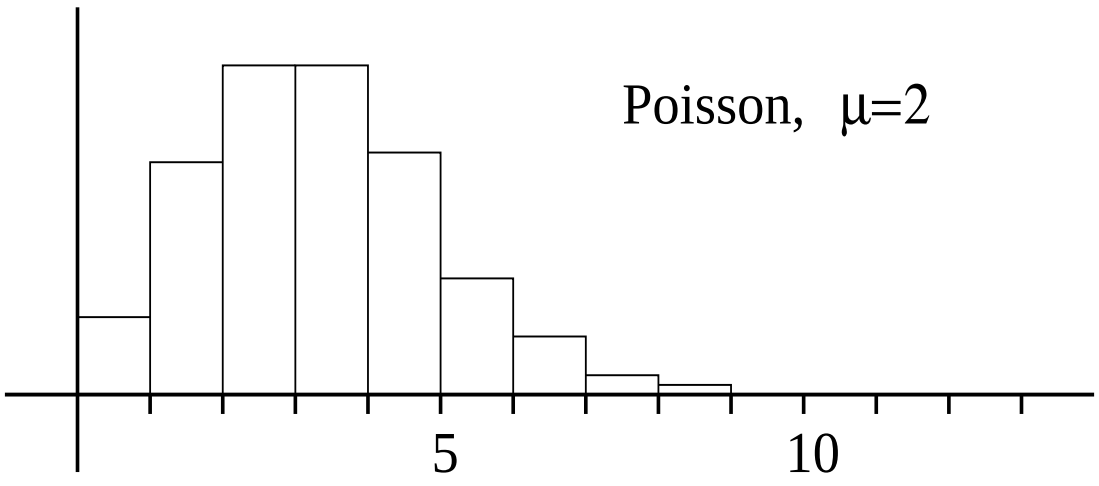
where $\mu = np$.

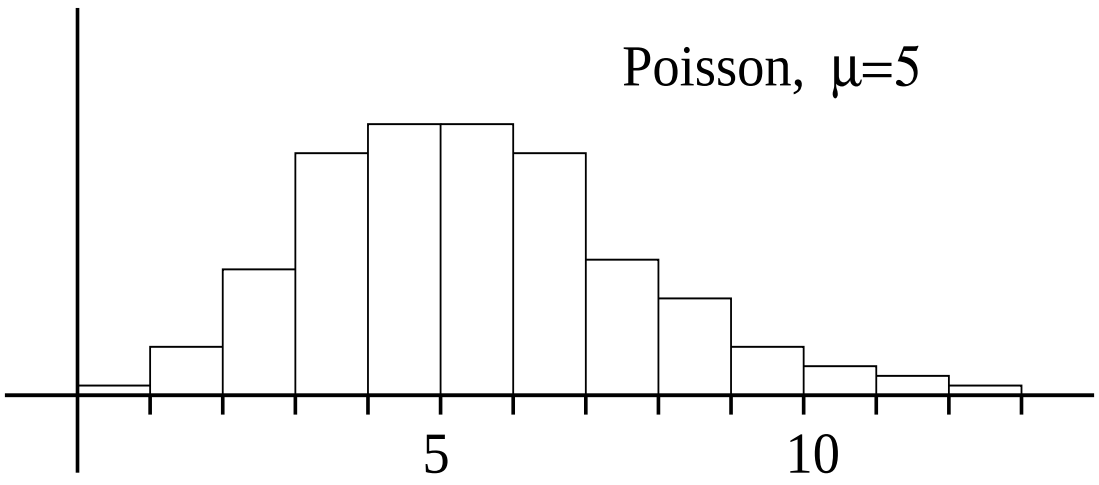
Poisson distribution

Take $p_k \;\equiv\; \frac{e^{-\mu}\mu^k}{k!}$, $k \geq 0$, where $\mu > 0$. Then

$$\sum_{k=0}^{\infty} p_k \;=\; e^{-\mu} \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \;=\; e^{-\mu}\, e^{\mu} \;=\; 1$$

$\Rightarrow$ the p_k define a set of probabilities!


Poisson, μ=2
5
10


Poisson, μ=5
5
10

Mode of Poiss(μ) distribution

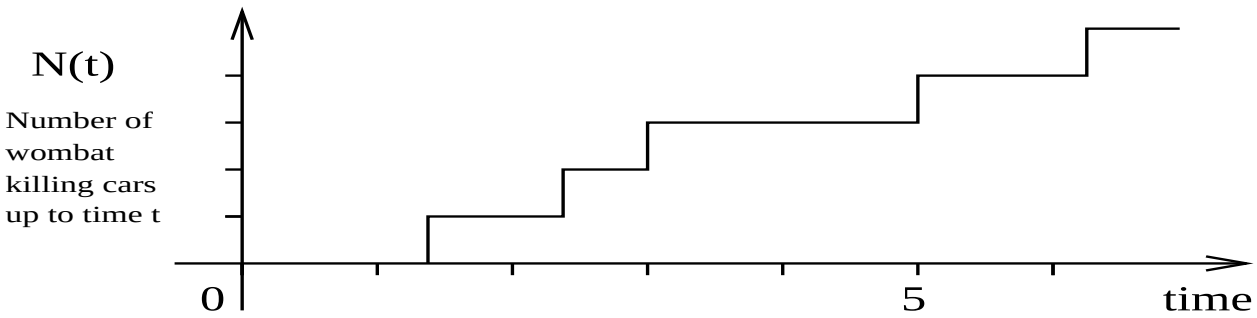
- We are looking for the <u>value</u> with the <u>largest</u> probability; i.e. the k^* for which

$$P\{k^*-1\} \leq P\{k^*\} \quad \text{and} \quad \leq P\{k^*\} \geq P\{k^*+1\}.$$

$$\begin{aligned} P\{k-1\} \leq P\{k\} \quad &\Longleftrightarrow \quad \frac{e^{-\mu}\mu^{k-1}}{(k-1)!} \leq \frac{e^{-\mu}\mu^{k}}{k!} \\ &\Longleftrightarrow \quad 1 \leq \frac{\mu}{k} \\ &\Longleftrightarrow \quad k \leq \mu \end{aligned}$$

$\Rightarrow$ the mode k^* satisfies $\mu \leq k^* \leq \mu + 1$, or, <u>if k is not an integer</u>, $k^* = [\mu + 1]$, the "largest integer less than or equal to $\mu + 1$".

Poisson Process



The process and its axioms

- $N(t)$ is the number of arrivals up to time t, $(N(0) \equiv 0)$.
- $s < t \leq u < v \quad \Rightarrow \quad N(v) - N(u)$ and $N(t) - N(s)$ are independent.
- It is "unlikely" that there will be ≥ 2 arrivals in a short period.
- There is an arrival rate $\lambda > 0$.

Formal axioms re arrivals

- Assume, for δ small,

(1) $$P\{N(t+\delta) = k+1 \mid N(t) = k\} \;=\; \lambda\delta \;+\; o(\delta),$$

where

$$g(x) = o(x) \;\Rightarrow \lim_{x\to 0} \frac{g(x)}{x} = 0.$$

(2) $$P\{N(t+\delta) > k+1 \mid N(t) = k\} \;=\; o(\delta),$$

which imply

(3) $$P\{N(t+\delta) = k \mid N(t) = k\} \;=\; 1 - \lambda\delta - o(\delta),$$

All of which imply

$$P\{N(t) = k\} \;=\; e^{-\lambda t}(\lambda t)^k/k!$$

i.e. $N(t)$ has a Poisson distribution with $\mu = \lambda t$.

Proof: Set $p_t(k) \;=\; P\{N(t) = k\}$.

$$
\begin{aligned}
p_{t+\delta}(k) \;&=\; p_t(k) \times P\{\text{No arrivals in } [t, t+\delta]\} \\
&\qquad + p_t(k-1) \times P\{1 \text{ arrival in } [t, t+\delta]\} \\
&\qquad\qquad + P\{\text{All other possibilities}\} \\
&=\; p_t(k)\,(1 - \lambda\,\delta - o(\delta)) \;+\; p_t(k-1)\,(\lambda\,\delta + o(\delta)) \;+\; o(\delta)
\end{aligned}
$$

Rearranging gives

$$
\frac{p_{t+\delta}(k) - p_t(k)}{\delta} \;=\; \lambda\, p_t(k) \;+ \lambda\, p_t(k-1) \;+\; \frac{o(\delta)}{\delta}.
$$

Sending $\delta \to 0$ gives

$$
\frac{d}{dt} p_t(k) \;=\; \lambda\, p_t(k) \;+ \lambda\, p_t(k-1),
$$

which is solved by

$$
p_t(k) \;=\; e^{-\lambda t} (\lambda t)^k / k!
$$

WOMBATS

$$\boxed{P\{N(t) = k\} \;=\; e^{-\mu}\mu^k/k!}$$

Example 1: Wombats A very old, solid, wombat stands at the side of the Trans Australian highway, at a point at which cars pass at the rate of two per hour.

- If the wombat waits for an hour, what is the probability that he will see exactly 4 cars?

  - $\mu \;=\; \lambda t \;=\; 2 \times 1 \;=\; 2.$
  - $P\{N(1) = 4\} \;=\; e^{-2}2^4/4! \;=\; 0.0902$

- It takes the wombat 45 minutes to cross the road, and three car collisions to kill him. What is the probability that he will make it across alive?

  - $\mu \;= \lambda t \;=\; 2 \times 0.75 \;=\; 1.5.$
  - $P\{\text{Survival}\} = P\{N(0.75) \leq 2\} = e^{-1.5}[1 + \frac{1.5}{1} + \frac{(1.5)^2}{2}] = 0.81$

$$\boxed{P\{N(t) = k\} \;=\; e^{-\mu}\mu^k/k!}$$

- Suppose T is the amount of time the wombat waits to see a car. How large is T likely to be?

$$P\{T \geq t\} \;=\; P\{N(t) = 0\} \;=\; e^{-\lambda t} \;=\; e^{-2t}.$$

- Suppose the wombat has already waited t hours to see a car, <u>without success</u>. What is the probability that he will have to wait <u>another</u> s hours?

$$\begin{aligned} P\{T > t + s \mid T > t\} &= \frac{P\{T > t + s \text{ and } T > t\}}{P\{T > t\}} \\ &= P\{T > t + s\} \;/\; P\{T > t\} \\ &= e^{-\lambda(t+s)} \;/\; e^{-\lambda t} \\ &= e^{-\lambda s} \\ &= P\{T > s\} \end{aligned}$$

Binomial distribution

n tosses of independent coins with probability p of success

Probabilities:

$$P\{k\} = \binom{n}{k} p^k q^{n-k}$$

~ Poiss(np), n large, p small

Mode = [np+p] ~ np

Poisson distribution

Number of arrivals at rate λ with independence between time periods

Probabilities:

$$P\{k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Mode = [λt+1]~λt

Geometric distribution

Number of tosses until a success exhibits lack of memory

Exponential distribution

Time until arrival exhibits lack of memory

Usage

Problem	Poisson?	Parameter (λ)	Parameter (μ)
No. of cars on Herzl crossing Balfour	No		
Telephone calls to exchange in an hour	Yes	Rate/hour λ	λ
Telephone calls to exchange in 30 mins	Yes	Rate/hour λ	$\lambda/2$
Telephone calls to exchange in 2 hours	Yes	Rate/hour λ	2λ
Telephone calls to 2 exchanges in 1 hour	Yes	Rates/hour $\lambda_1,\ \lambda_2$	$\lambda_1 + \lambda_2$

Strontium 90 (p282)

- Radioactive decay is a Poisson process: λ=?
- "Half life" of material = 28 years
- $\Rightarrow$ "Half life" of <u>nuclei</u> = 28 years

$$1/2 \;=\; P\{\text{Time to decay of } \underline{\text{specific}} \text{ nucleus} > 28\} \;=\; e^{-28\lambda}$$

$$\Rightarrow \lambda \;=\; \frac{\log(1/2)}{-28} \;=\; 0.0248 \text{ per year.}$$

Example: What is the probability that of 5 specific nuclei, at least 3 decay within 2 years?

- $\mu \;=\; 5 \times 2 \times 0.0248 \;=\; 0.248$

$$\begin{aligned} P\{\geq 3 \text{ decay}\} &= 1 - P\{< 3 \text{ decay}\} \\ &= 1 - e^{-0.248}[1 + 0.248 + 0.248^2/2] \;=\; .0002 \end{aligned}$$