Lecture 34

Conditional variance

Sums of discrete random variables

Sums via moment generating functions

Convolution formula for densities

Examples

Conditional variance

- X and Y are either discrete or continuous.
- $\mu_X = E(X), \quad \sigma_X^2 = Var(X)$
- $\mu_Y = E(Y), \quad \sigma_Y^2 = Var(Y)$
- $\mu_{X|Y} = E(X|Y), \quad \mu_{Y|X} = E(Y|X)$
- $\mu_Y = E(\mu_{Y|X})$
- Define conditional variance by

$$\sigma_{Y|X}^2 \;=\; E\left\{(Y - \mu_{Y|X})^2 \middle| X\right\}.$$

- Is it true that $\sigma_Y^2 = E\{\sigma_{Y|X}^2\}$, and, if not, what is true?

Conditional variance: Con't

$$
\begin{aligned}
\sigma_Y^2 &= E\left[(Y-\mu_Y)^2\right] \\
&= E\left\{E\left[(Y-\mu_Y)^2 \;\middle|\; X\right]\right\} \\
&= E\left\{E\left[\left((Y-\mu_{Y|X}) \;+\; (\mu_{Y|X}-\mu_Y)\right)^2 \;\middle|\; X\right]\right\} \\
&= E\left\{E\left[\left(Y-\mu_{Y|X}\right)^2 \;\middle|\; X\right]\right\} + E\left\{E\left[\left(\mu_{Y|X}-\mu_Y\right)^2 \;\middle|\; X\right]\right\} \\
&\qquad\qquad + 2E\left\{E\left[\left(Y-\mu_{Y|X}\right)\cdot\left(\mu_{Y|X}-\mu_Y\right) \;\middle|\; X\right]\right\} \\
&= E(\sigma_{Y|X}^2) \;+\; Var(E(Y|X))
\end{aligned}
$$

- In words

$$Var(Y) \;=\; E(\text{Conditional variance of } Y|X) \;+\; Var(E(Y|X))$$

Discrete Uniform-Binomial

- $N+1$ boxes, labelled $0, 1, 2, \ldots, N$. Box number b has b black balls and $N-b$ white balls.
- Choose a box at random, with number B, and draw n balls, with replacement. S_n is the total number of black balls.

$$E(S_n) = E[E(S_n|B)] = E[n \cdot B/N] = \frac{n}{N} E[B] = \frac{n}{2}$$

$$\begin{aligned} Var(S_n) &= E(\sigma^2_{S_n|B}) + Var(E(S_n|B)) \\ &= E\left[n \frac{B}{N} \frac{N-B}{N}\right] + Var(n \cdot B/N) \\ &= \frac{n}{N^2}\left[\frac{N \cdot N}{2} - \frac{N(2N+1)}{6}\right] + \left(\frac{n}{N}\right)^2 \frac{N^2+2N}{12} \end{aligned}$$

- Note second term is dominant
- $Var(S_n) \to n^2/12$ as $N \to \infty$, and also as $n \to \infty$.

Sums of discrete random variables

- X and Y discrete, $p_{XY}(x,y) = P\{X = x, Y = y\}$
- $Z = X + Y$

$$\begin{aligned} p_Z(z) &= \sum_x P\{Z = z, X = x\} \\ &= \sum_x P\{X + Y = z, X = x\} \\ &= \sum_x P\{Y = z - x, X = x\} \\ &= \sum_x P\{Y = z - x | X = x\} P\{X = x\} \end{aligned}$$

- X, Y independent gives the discrete convolution formula:

$$p_Z(z) = \sum_x P\{Y = z - x\} P\{X = x\} = \sum_x p_Y(z - x) \cdot p_X(x)$$

Sum of discrete uniforms

- X, Y independent, $U[1, N]$

$$p_{X+Y}(z) \quad = \quad \sum_x p_Y(z-x) \cdot p_X(x)$$

- Note that $X + Y \in [2, 2N]$, but $p_Y(y)$ and $p_X(x)$ are non-zero only for $x, y \in [1, N]$.

CASE 1: $2 \leq z \leq N + 1$

$$p_{X+Y}(z) \;=\; \sum_{x=1}^{z-1} 1/N^2 \;=\; (z-1)/N^2$$

CASE 2: $N + 1 \leq z \leq 2N$

$$p_{X+Y}(z) \;=\; \sum_{x=z-N}^{N} 1/N^2 \;=\; (2N-z)/N^2$$

Sums via moment generating functions

- Moment generating function

$$M_X(t) \;=\; E\left\{e^{tX}\right\} \;=\; \sum_x e^{tx}\; p_X(x) \;\;\text{OR}\;\; \int e^{tx} f_X(x)\; dx$$

- $X_1, \ldots, X_n$ <u>independent</u> with MGF's $M_{X_1}(t), \ldots, M_{X_n}(t)$.

- $S_n = X_1 + \ldots + X_n$

$$M_{S_n}(t) \;=\; M_{X_1}(t) \cdots M_{X_n}(t)$$

Convolution formula for densities

- (X, Y) has joint density $f_{XY}(x, y)$
- $Z = X + Y$

$$\boxed{\begin{aligned} f_{X+Y}(z) &= \int f_{XY}(x, z-x)\, dx \\ &\overset{IND}{=} \int f_X(x) f_Y(z-x)\, dx \end{aligned}}$$

- Proof 1: Via distribution functions
- Proof 2: Via infinitessimals

Sum of continuous uniforms

- X, Y independent, $U[0,1]$

$$f_{X+Y}(z) \quad = \quad \int f_X(x) f_Y(z-x)\ dx$$

- Note that $X+Y \in [0,2]$, but $f_Y(y)$ and $f_X(x)$ are non-zero only for $x, y \in [0,1]$.

CASE 1: $0 \leq z \leq 1$

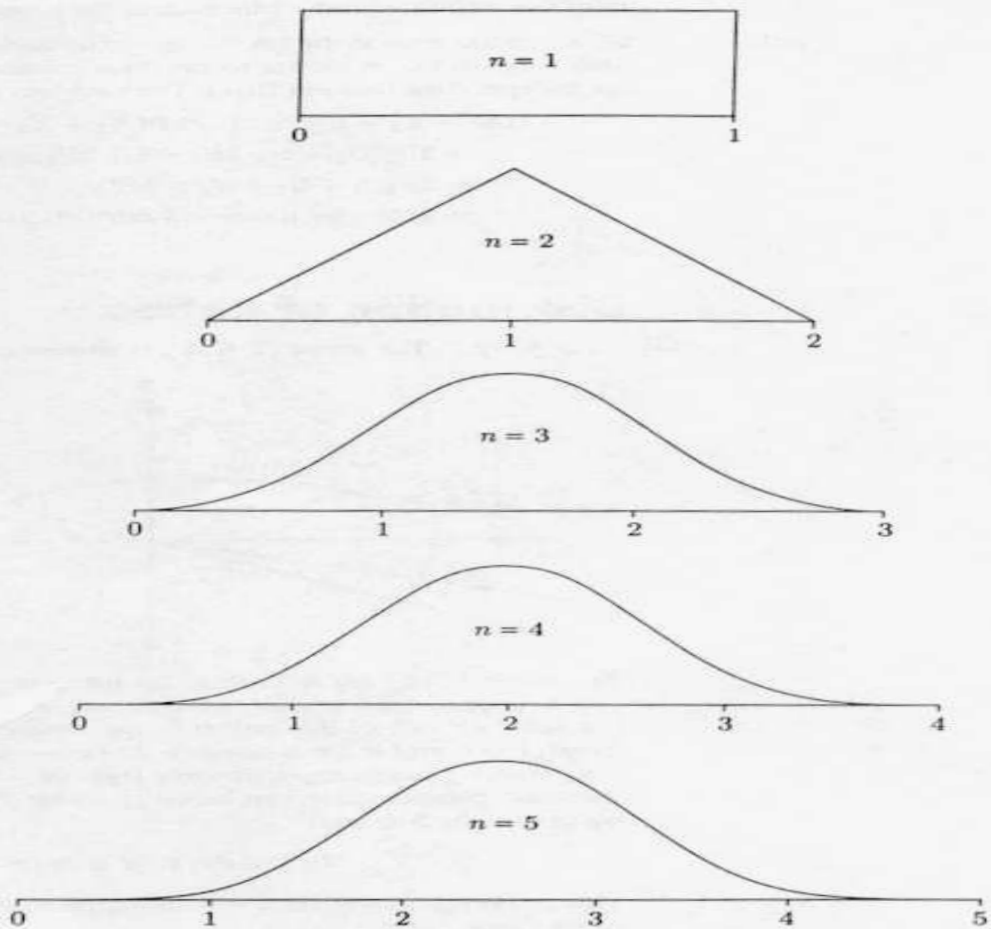
$$f_{X+Y}(z) \ = \ \int_{x=0}^{z} 1\ dx \ = \ z$$

CASE 2: $1 \leq z \leq 2$

$$f_{X+Y}(z) \ = \ \int_{x=z-1}^{1} 1\ dx \ = \ (2-z)$$

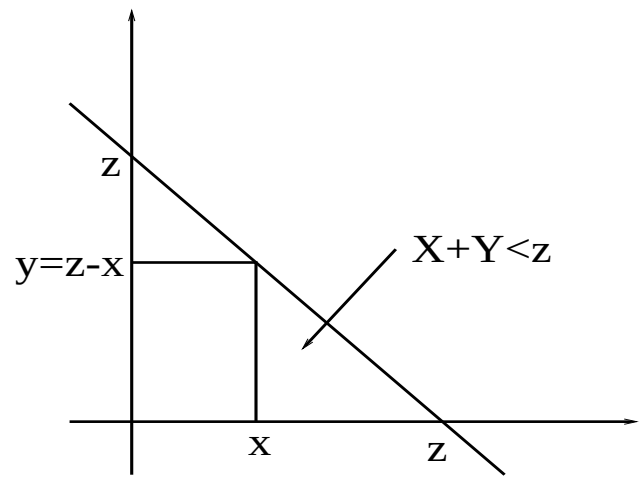
Sums of continuous uniforms

FIGURE 4. **Density of the sum of n independent uniform $(0, 1)$ variables.** The graphs are all centered at the mean with a constant horizontal distance on the page representing one standard unit in each graph. This shows how rapidly the shape of the distribution becomes normal as n increases.



Proof 1: Via distribution functions

$$
\begin{aligned}
&F_{X+Y}(z) \\
&= P\{X+Y \le z\} \\
&= \int_{\text{Lower half plane}} f_{XY}(x,y)\ dxdy \\
&= \int_x \int_{-\infty}^{z-x} f_{XY}(x,y)\ dxdy \\
&= \int_x \left(\int_{-\infty}^{z-x} f_{XY}(x,y)\ dy \right)\ dx
\end{aligned}
$$

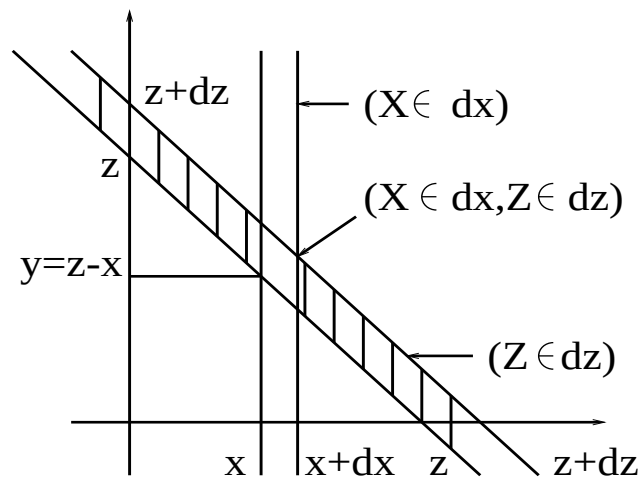


- Recall: $f_Z(z) = d/dz(F_Z(z))$.

$$
f_{X+Y}(z) = \int_x \frac{d}{dz} \left(\int_{-\infty}^{z-x} f_{XY}(x,y)\ dy \right)\ dx = \int_x f_{XY}(x, z-x)\ dx
$$

Proof 2: Via infinitessimals

$$
\begin{aligned}
& f_{X+Y}(z)\ dz \\
&= P\{X+Y \in dz\} \\
&= \int_{\text{Hatched area}} f_{XY}(x,y)\ dxdy \\
&= \int_x \int_{z-x}^{z-x+dz} f_{XY}(x,y)\ dxdy \\
&= \int_x \left(f_{XY}(x, z-x)\ dz\right)\ dx \\
&= \left(\int_x f_{XY}(x, z-x)\ dx\right)\ dz
\end{aligned}
$$



Sums of exponentials

- $X, Y \sim \exp(\lambda)$, independent. $Z = X + Y$
  (Already done via moment generating functions)

$$
\begin{aligned}
f_{X+Y}(z) &= \int f_X(x) f_Y(z-x)\, dx \\
&= \int_0^z \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)}\, dx \\
&= \lambda^2 e^{-\lambda z} \int_0^z dx \\
&= \lambda^2 z e^{-\lambda z} \qquad\qquad z \geq 0
\end{aligned}
$$

- i.e. $X + Y \sim \text{Gamma}(2, \lambda)$.

- Similarly, or via MGF's, or Poisson process can show
  $\sum_k \text{Gamma}(n_k, \lambda) \;\sim\; \text{Gamma}(\sum_k n_k, \lambda)$

Sums of Gamma distributions

$$X \sim \text{Gamma}(r, \lambda) \iff f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, \quad x > 0$$

- $X \sim \text{Gamma}(r, \lambda)$, $Y \sim \text{Gamma}(s, \lambda)$, independent.
- We will show that $X + Y \sim \text{Gamma}(r + s, \lambda)$, for any $r > 0$ and $s > 0$.
- Note: This <u>cannot</u> be expressed in terms of wombats.
- Recall:

$$\begin{aligned} X &\sim \text{Beta}(r, s) \\ \iff f_X(x) &= \frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1} \\ &= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}, \qquad 0 < x < 1 \end{aligned}$$

$$
\begin{aligned}
f_{X+Y}(z) &= \int_x f_X(x)\ f_Y(z-x)\ dx \\
&= \int_0^z \frac{1}{\Gamma(r)}\lambda^r x^{r-1} e^{-\lambda x} \cdot \frac{1}{\Gamma(s)}\lambda^s (z-x)^{s-1} e^{-\lambda(z-x)}\ dx \\
&= \int_0^z \frac{1}{\Gamma(r)\Gamma(s)}\ \lambda^{r+s}\ x^{r-1}(z-x)^{s-1}\ e^{-\lambda z}\ dx \\
&= \int_0^1 \frac{1}{\Gamma(r)\Gamma(s)}\ \lambda^{r+s}\ (zu)^{r-1}(z-zu)^{s-1}\ e^{-\lambda z}\ z\ du \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x = zu,\ dx = z\ du) \\
&= \frac{1}{\Gamma(r+s)}\lambda^{r+s}\ z^{r+s-1}\ e^{-\lambda z} \\
&\qquad\qquad \times \int_0^1 \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\ u^{r-1}(1-u)^{s-1}\ du \\
&= \frac{1}{\Gamma(r+s)}\lambda^{r+s}\ z^{r+s-1}\ e^{-\lambda z}
\end{aligned}
$$