Lecture 6

Generating non-uniform spaces

Multiplication rule

Tossing biased coins

Geometric Distribution

Independence

Sequences of events

- Tossing a coin n times.
- Tossing a coin until a H is seen.
- Choosing 7 balls from one of 8 boxes, with 0–8 white balls per box.

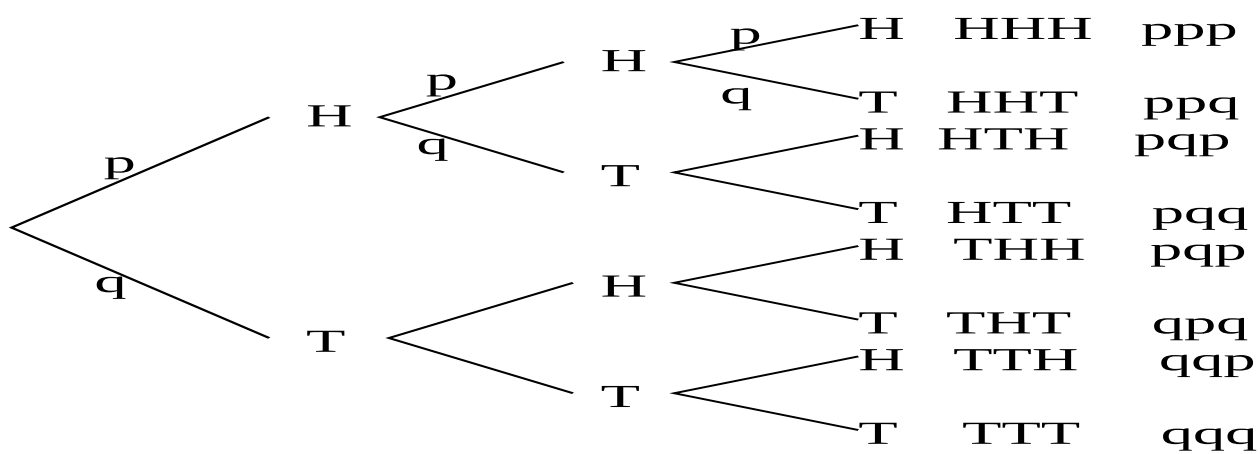
Multiplication rule for n events:

$$P\{A_1 A_2 \ldots A_n\}$$
$$= P\{A_1\}P\{A_2|A_1\}P\{A_3|A_1A_2\} \times \cdots P\{A_n|A_1A_2\ldots A_{n-1}\}$$

No. of heads

3 biased coins

- $P\{\text{Head}\} = p, \quad P\{\text{Tail}\} = q, \quad (p + q = 1)$



No. of heads	0	1	2	3
Probability	q^3	$3q^2p$	$3qp^2$	p^3

Tossing for a head

Biased coin

- Probability that it takes <u>3 or less</u> tosses to see a head?

$$\begin{aligned} P\{3 \text{ or less}\} &= P\{H, TH, TTH\} \\ &= P\{H\} + P\{TH\} + P\{TTH\} \;=\; p + qp + qqp. \end{aligned}$$

- Probability that it takes <u>exactly 3</u> tosses to see a head?

$$P\{\text{Exactly 3 tosses}\} \;=\; P\{TTH\} \;=\; qqp$$

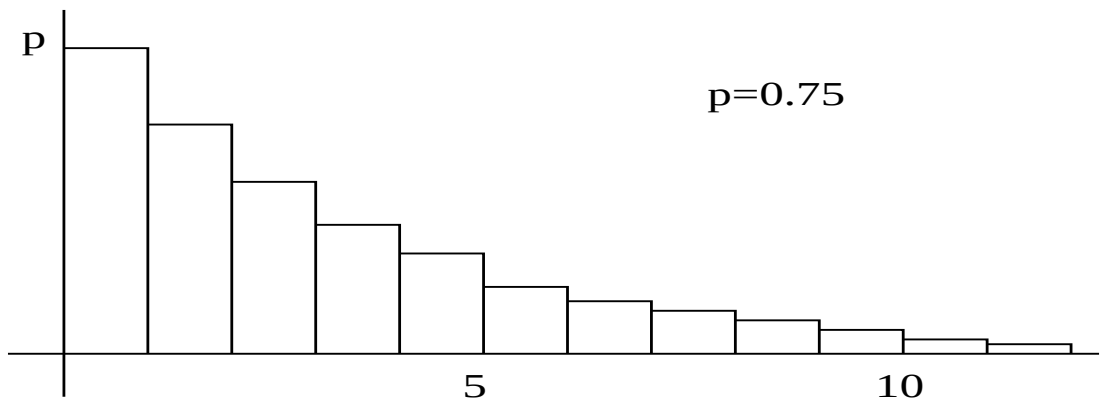
- Probability that it takes <u>exactly k</u> tosses to see a head?

$$P\{\text{Exactly k tosses}\} \;=\; P\{\overbrace{TT\ldots T}^{k-1}H\} \;=\; q^{k-1}p$$

Geometric Distribution

- If X is the number of tosses to see the <u>first</u> head in successive tosses of a coin, then, for all $k \geq 1$,

$$\boxed{P\{X = k\} \;=\; pq^{k-1}}$$



- X is called a geometric random variable.
- Note: $\sum_{k=1}^{\infty} pq^{k-1} \equiv 1 \;\Rightarrow$

$$\frac{1}{1-q} \;=\; \frac{1}{p} \;=\; 1 + q + q^2 + q^3 + \ldots$$

Lack of memory

- If X has a Geom(p) distribution, what is (for $k \geq 1$)

$$P\{X \geq n+k \mid X > n\}?$$

$$\frac{P\{X \geq n+k \text{ and } X > n\}}{P\{X > n\}} = \frac{P\{X \geq n+k\}}{P\{X > n\}}$$

$$P\{X \geq n\} = \sum_{k=n}^{\infty} P\{X = k\} = \sum_{k=n}^{\infty} pq^{k-1} = q^{n-1} \sum_{k=1}^{\infty} pq^{k-1} = q^{n-1}$$

$$P\{X \geq n+k \mid X > n\} = \frac{q^{n+k-1}}{q^{(n+1)-1}} = q^{k-1} \equiv P\{X \geq k\}$$

Negative binomial distribution

- Toss a coin for which $P\{\text{Success}\} = p$
- X = the number of tosses until the the r-th success.

$$\underbrace{SSFF\ldots\ldots\ldots\ldots\ldots\ldots\ldots FFSS}_{x-1 \text{ tosses, with } r-1 \text{ successes}}\ S$$

$$\Rightarrow P\{X = x\} \;=\; \binom{x-1}{r-1} p^r q^{x-r}, \qquad x \geq r$$

Banach's matchbox problem

- Drawing matches at random from two boxes with N matches, what the probability that there will be M left in the non-empty one when one box is found to be empty?

- Pattern to find box #1 empty with M in box #2:

$$\underbrace{1122\ldots\ldots\ldots\ldots\ldots\ldots\ldots 212}_{2N-M \text{ choices, with } N \text{ 1's}}\ 1$$

- This is negative binomial, with $x = 2N - M + 1$ and $r = N + 1$ and so the solution is

$$2 \times \binom{2N-M}{N} \left(\frac{1}{2}\right)^{2N-M+1}, \qquad 0 \le M \le N.$$

Independence

- $A_1, A_2, \ldots A_n$ are <u>independent</u> if, and only if, for all appropriate subsets of $\{1, 2, \ldots, n\}$

$$(1) \quad P\{A_i\} = P\{A_i|A_j\} = P\{A_i|A_j^c\}$$

$$(2) \quad P\{A_i\} = P\{A_i|A_j A_k\}$$

$$= P\{A_i|A_j^c A_k\} = P\{A_i|A_j A_k^c\} = P\{A_i|A_j^c A_k^c\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(n-1) \quad P\{A_i|A_1 \ldots \underbrace{A_i}_{\text{missing}} \ldots A_n\} = \ldots = P\{A_i\}$$

- Thus $A_1, A_2, \ldots A_n$ are <u>independent</u> if, and only if, for all collections $\{i_1, \ldots, i_k\}$ and all $k \in \{2, \ldots, n\}$

$$P\{A_{i_1} \ldots A_{i_k}\} \;=\; P\{A_{i_1}\} \ldots P\{A_{i_k}\}$$

Proof: Start with $k = 2$ (given) and then iterate upwards.

- $\Rightarrow$ need to check many combinations:

$$\binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} \;=\; 2^n - n - 1$$

Proof:

$$2^n \;=\; (1+1)^n \;=\; \sum_{k=0}^{n} \binom{n}{k}$$

Independence in pairs is not enough!

- Toss two fair coins and let

  - $H_1 = \{\text{Head on first toss}\} = \{HH, HT\} \Rightarrow \; P\{H_1\} = 1/2$
  - $H_2 = \{\text{Head on 2nd toss}\} = \{TH, HH\} \Rightarrow P\{H_2\} = 1/2$
  - $S = \{\text{Both tosses the same}\} = \{HH, TT\} \Rightarrow \; P\{S\} = 1/2$

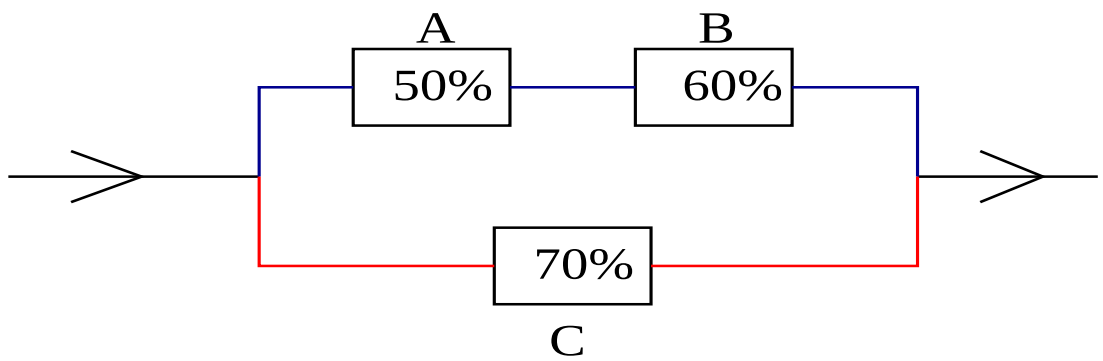
- Then

$$P\{H_1 \cap H_2\} = P\{HH\} = 1/4 = P\{H_1\}P\{H_2\}$$
$$P\{H_j \cap S\} = P\{HH\} = 1/4 = P\{H_j\}P\{S\}$$

BUT

$$P\{H_1 \cap H_2 \cap S\} = P\{HH\} = 1/4 \neq 1/8 = P\{H_1\}P\{H_2\}P\{S\}$$

Another "flow" problem



- A works 50% of the time, B 60%, and C 70%.
  All components are independent.
  What is the probability that there is an "open path"?

$$
\begin{aligned}
P\{\text{Path}\} &= P\{\text{Upper open}\} + P\{\text{Lower open}\} - P\{\text{Both open}\} \\
&= 0.5 \times 0.6 \ + \ 0.7 \ - \ 0.5 \times 0.6 \times 0.7 \\
&= 0.79
\end{aligned}
$$

- If the system has failed, what is the probability that A is still functional?

How many games?

- I play a game, with probability α (<u>small</u>) of winning. How many <u>independent</u> games, n, do I need to play to be at least 50% certain of winning <u>at least</u> once?

$$P\{\geq 1 \text{ win in } n \text{ games}\} = 1 - P\{0 \text{ wins}\} = 1 - (1-\alpha)^n.$$

We want the least n for which

$$1 - (1-\alpha)^n > 1/2 \;\Leftrightarrow\; (1-\alpha)^n < 1/2 \;\Leftrightarrow\; n\ln(1-\alpha) < \ln(1/2)$$

<u>Guess</u> that the solution $n^* \sim c/\alpha$ and use $\ln(1+x) \sim x$ for small x:

$$\ln(1/2) \sim \frac{c}{\alpha}\ln(1-\alpha) \sim \frac{c}{\alpha}(-\alpha) = -c$$

- Need to play about $\frac{\ln 2}{\alpha} \sim \frac{2}{3\alpha}$ games.