Lecture 27

Maxima and minima

Inverse DF

Percentiles

Simulations

Maximum and Minimum

- $X_1, \ldots, X_n$ <u>independent</u> random variables with distribution functions $F_1, \ldots, F_n$.
- $X_{\max} \equiv \max(X_1, \ldots, X_n), \quad X_{\min} \equiv \min(X_1, \ldots, X_n)$

$$\begin{aligned} F_{\max}(x) \equiv P\{X_{\max} \le x\} &= P\{X_1 \le x, \ldots, X_n \le x\} \\ &= P\{X_1 \le x\} \cdots P\{X_n \le x\} \\ &= F_1(x) \cdots F_n(x) \end{aligned}$$

$$\begin{aligned} F_{\min}(x) \equiv P\{X_{\min} \le x\} &= 1 - P\{X_{\min} > x\} \\ &= 1 - P\{X_1 > x, \ldots, X_n > x\} \\ &= 1 - P\{X_1 > x\} \cdots P\{X_n > x\} \\ &= 1 - (1 - F_1(x)) \cdots (1 - F_n(x)) \end{aligned}$$

Minimum of exponentials is exponential.

Suppose the X_1 are all $\exp(\lambda_i)$. Then

$$F_i(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda_i x} & \text{if } x \geq 0 \end{cases}$$

Clearly, $F_{\min}(x) = 0$ if $x < 0$. For $x \geq 0$,

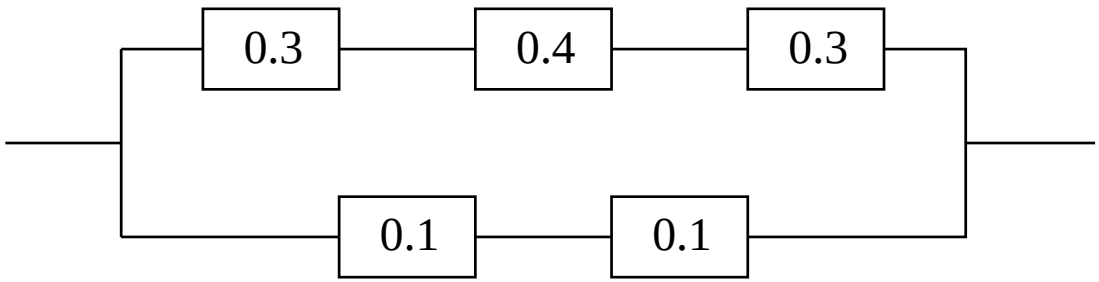
$$\begin{aligned} F_{\min}(x) &= 1 - e^{-\lambda_1 x} \cdots e^{-\lambda_n x} \\ &= 1 - e^{-(\lambda_1 + \ldots + \lambda_n)x} \end{aligned}$$

so that $X_{\min} \sim \exp(\lambda_1 + \ldots + \lambda_n)$.

- No similar result is true for $X_{\max}$.

Expected lifetime

- Circuit, with components living for an exponential time, with parameters as shown.



$$\text{Lifetime} \;=\; L \;=\; \max\Big(L_{\text{top}}, L_{\text{bottom}}\Big)$$

- From last slide, we know

$$\begin{aligned} L_{\text{top}} &\sim \exp(0.3 + 0.4 + 0.3) \;=\; \exp(1) \\ L_{\text{bottom}} &\sim \exp(0.1 + 0.1) \;=\; \exp(0.2) \end{aligned}$$

and so

$$\begin{aligned} F_L(x) &= F_{L_{\text{top}}}(x) F_{L_{\text{bottom}}}(x) \\ &= \left(1 - e^{-x}\right) \left(1 - e^{-0.2x}\right) \\ &= 1 - e^{-x} - e^{-0.2x} + e^{-1.2x} \end{aligned}$$

Now using (from the last lecture) the fact that

$$E\{X\} = \int_0^\infty (1 - F(x))\, dx$$

we have

$$E(L) = \int_0^\infty \left(e^{-x} - e^{-0.2x} + e^{-1.2x}\right) dx = 1 + (1/0.2) + (1/1.2) = 5.17$$

Alternatively, note

$$f_L(x) = F_L'(x) = e^{-x} + 0.2e^{-0.2x} - 1.2e^{-1.2x}$$

and integrate $\int_0^\infty x\ f_L(x)\ dx$.

Max and min of uniforms

- $X_1, \ldots, X_n$ independent, $U[0,1]$.

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Which implies, for $0 \leq x \leq 1$,

$$F_{\max}(x) = F^n(x) \;=\; x^n, \qquad f_{\max}(x) = n\; x^{n-1}$$

$$F_{\min}(x) = 1 - (1 - F(x))^n \;=\; 1 - (1-x)^n \qquad f_{\min}(x) = n\;(1-x)^{n-1}$$

- Expectations

$$E\{X_{\max}\} \;=\; \int_0^1 x\; n\; x^{n-1}\; dx \;=\; \frac{n}{n+1}$$

$$E\{X_{\min}\} \;=\; \int_0^1 x\; n\; (1-x)^{n-1}\; dx \;=\; \frac{1}{n+1}$$

Asymptotics

- $E\{X_{\min}\} = 1/(n+1)$
- $E\{X_{\min}^2\} \;=\; \int_0^1 x^2 \; n \; (1-x)^{n-1} \; dx \;=\; 2/[(n+1)(n+2)]$
- $$\begin{aligned} Var(X_{\min}) \;&=\; 2/[(n+1)(n+2)] - 1/(n+1)^2 \\ &=\; n/[(n+2)(n+1)^2] \end{aligned}$$
- By Chebychev

$$\begin{aligned} P\left\{\left|X_{\min} - \frac{1}{n+1}\right| \;>\; \varepsilon\right\} \;&\leq\; \frac{n/(n+2)(n+1)^2}{\varepsilon^2} \\ &\rightarrow\; 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

Distributional (fine) asymptotics

- We now know that $X_{\min}$ is "close" to $1/(n+1)$ for large n.
- Consider

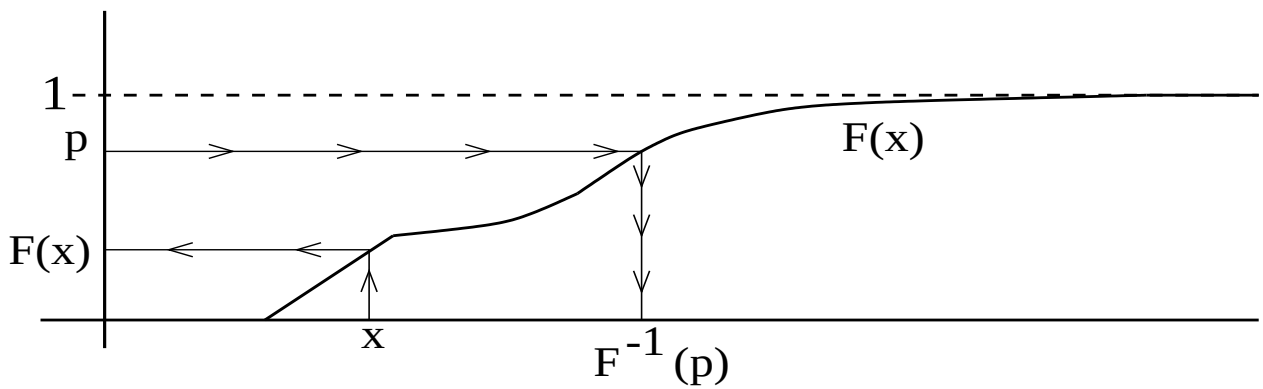
$$
\begin{aligned}
P\{X_{\min} \leq \frac{\alpha}{n}\} &= 1 - P\{X_{\min} \geq \frac{\alpha}{n}\} \\
&= 1 - P\{X_i \geq \frac{\alpha}{n},\ \forall i = 1, \ldots, n\} \\
&= 1 - (1 - \alpha/n)^n \\
&\rightarrow 1 - e^{-\alpha}
\end{aligned}
$$

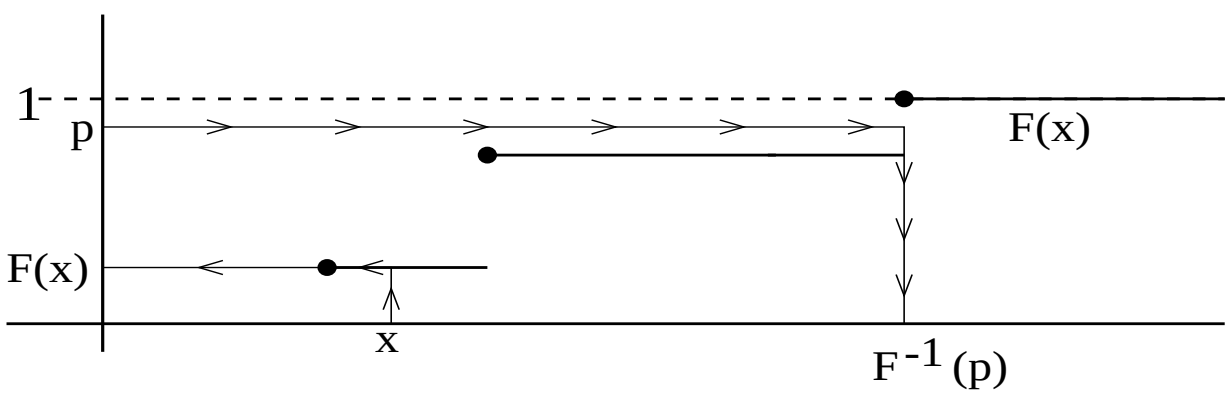
- i.e. $P\{nX_{\min} \leq \alpha\} \approx 1 - e^{-\alpha}$
- We say that $nX_{\min}$ has, "asymptotically", an exponential distribution

Inverse distribution function

$$F^{-1}(p) \equiv \inf\{x : F(x) \geq p\}$$

F continuous $\Rightarrow F^{-1}(p) \equiv x : F(x) = p$





Inverse DF: properties

- Defining <u>percentiles</u>: The n-th percentile is defined by $F^{-1}(n/100)$.

- Application to the standard uniform, $U \sim U[0,1]$

$$\begin{aligned} P\{F^{-1}(U) \leq x\} &= P\{F(F^{-1}(U)) \leq F(x)\} \\ &= P\{U \leq F(x)\} \\ &= F(x) \end{aligned}$$

since $P\{U \leq x\} = x, \quad x \in [0,1]$

- Which implies that the random variable $F^{-1}(U)$ has the distribution function F.

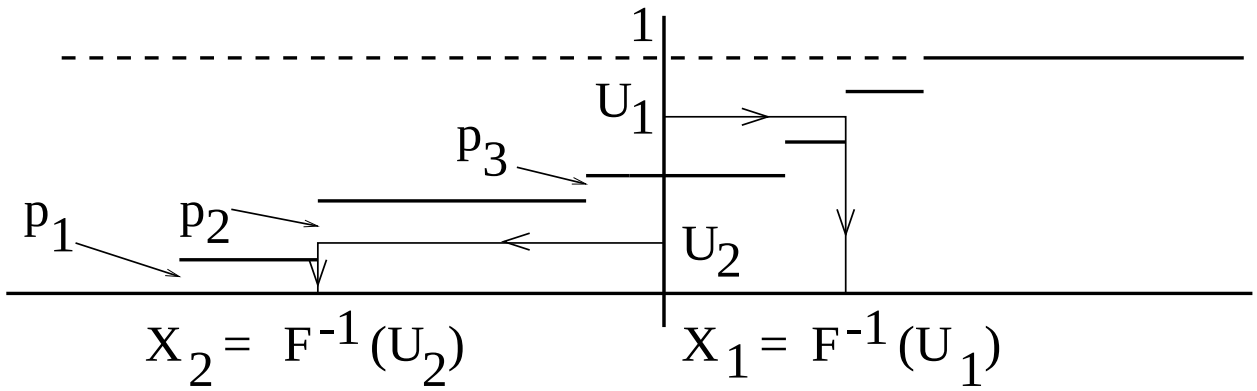
Simulation

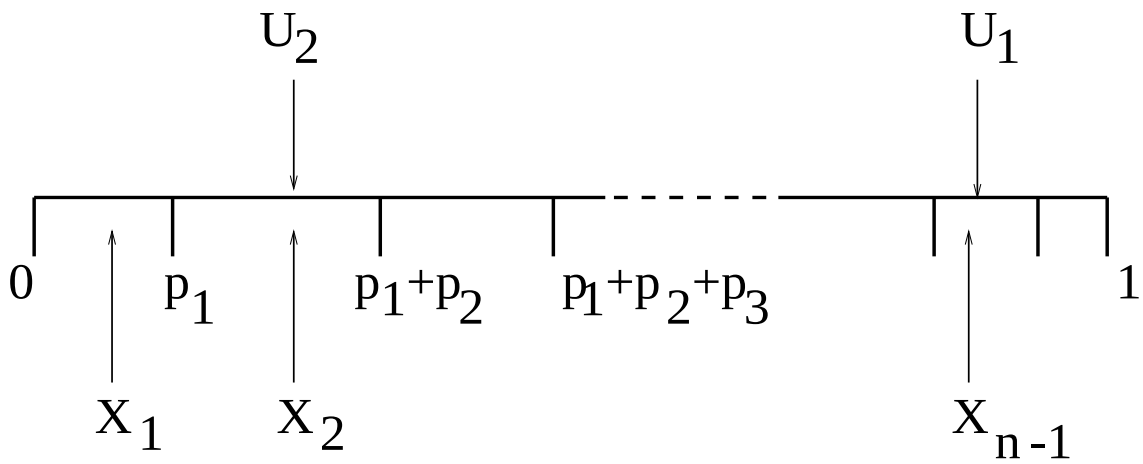
- Assume that we know how to simulate a $U \sim U[0,1]$ random variable.
- We want to simulate a random variable, X, with DF F.
- By the last slide, $X = F^{-1}(U)$ has the required DF.
- Example: $X \sim \exp(\lambda)$.

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \qquad F(x) = 1 - e^{-\lambda x}$$
$$\Rightarrow F^{-1}(p) = \log(1-p)/(-\lambda)$$
$$\Rightarrow X = \frac{-\log(1-U)}{\lambda} \quad \text{has an } \exp(\lambda) \text{ dist'n.}$$

- Note: This procedure will be <u>numerically</u> efficient only when $F^{-1}(p)$ is a simple function of p.

Simulating a discrete distribution





Algorithm for simulating a discrete distribution

- We want to simulate a random variable X taking the values $\{x_1, x_2, \ldots\}$ with probabilities $\{p_1, p_2, \ldots\}$
- Define

$$g(u) = \begin{cases} x_1 & \text{if } 0 \leq u \leq p_1 \\ x_k & \text{if } p_1 + \ldots + p_{k-1} < u \leq p_1 + \ldots + p_k, \text{and } k \geq 2 \end{cases}$$

- Take $U \sim U[0,1]$.
- Define $X = g(U)$.
- Then X takes the values $\{x_1, x_2, \ldots\}$ with probabilities $\{p_1, p_2, \ldots\}$.