Lecture 10

Random variables

Distributions

Functions of random variables

Random processes

Joint distributions

Random variables

- 1: The number of heads in n tosses of a coin.
- 2: The number of cars seen by a wombat.
- 3: The number of wombats killed in a day.
- 4: Time until wombat death.
- 5: The number of "good" (or "bad") items.
- 6: The numbers of items of various kinds.

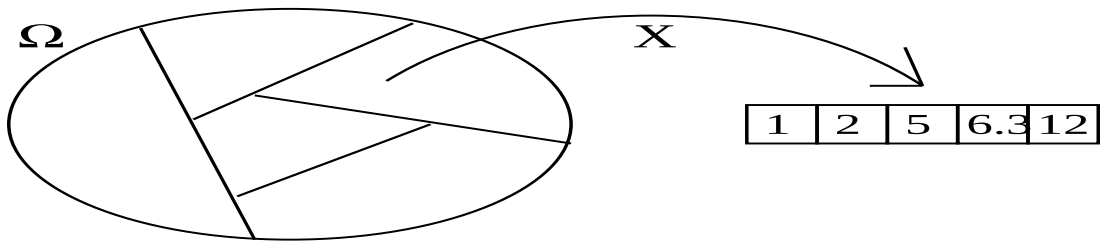
	1	2	3	4	5	6
Mode	Y	Y	y		y	
Mean (Expectation)	y	y	y	y	y	?
Dependence					Y	y

A random variable is a mapping
from a probability space to a set of values

$$\Omega = \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{bmatrix}$$

DISTRIBUTION OF SUM									
Sum $= X$	2	3	4	5	6	7	8	...	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	...	$\frac{1}{36}$

Definitions



- A random variable (rv) is a mapping from a probability space to a set of values.
- The range of a rv is the collection of all its possible values.
- The distribution of a rv is the collection of probabilities $\{P(A)\}$, where A ranges over all the subsets of the range.
- A discrete rv is one for which the range is countable.
- FACT: If X is a discrete random variable, then

$$P\{A\} = \sum_{x \in A} P\{X = x\}.$$

Binomial random variable:

$$X \sim \mathrm{Bin}(n,p) \iff P\{X=k\} = \begin{cases} \binom{n}{k} p^k q^{n-k} & \text{if } 0 \le k \le n \ , \\ 0 & \text{otherwise.} \end{cases}$$

- Range $= \{0, 1, \ldots, n\}$.
- $A = \{X \le 3\} \ \Rightarrow \ P\{A\} = \sum_{k=0}^{3} \binom{n}{k} p^k q^{n-k}$.
- $A = \{X \text{ is even}\} \ \Rightarrow \ P\{A\} = \sum_{k=0}^{[n/2]} \binom{n}{2k} p^{2k} q^{n-2k}$.

Poisson random variable:

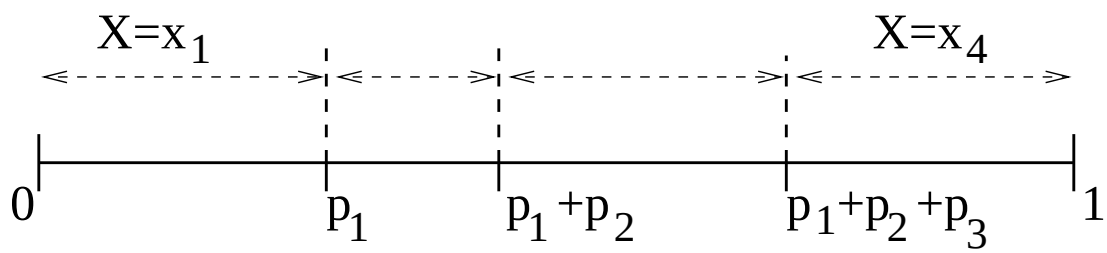
$$X \sim \mathrm{Pois}(\lambda) \iff P\{X=k\} = \frac{e^{-\lambda}\lambda^k}{k!}.$$

- Range $= \{0, 1, \ldots\}$.

The function $p_X(k) \equiv p(k) \equiv P\{X=k\}$ is called the <u>probability function</u> for the random variable X.

Random variables from the uniform, or Existence of $(\Omega, \mathcal{F}, \mathcal{P})$

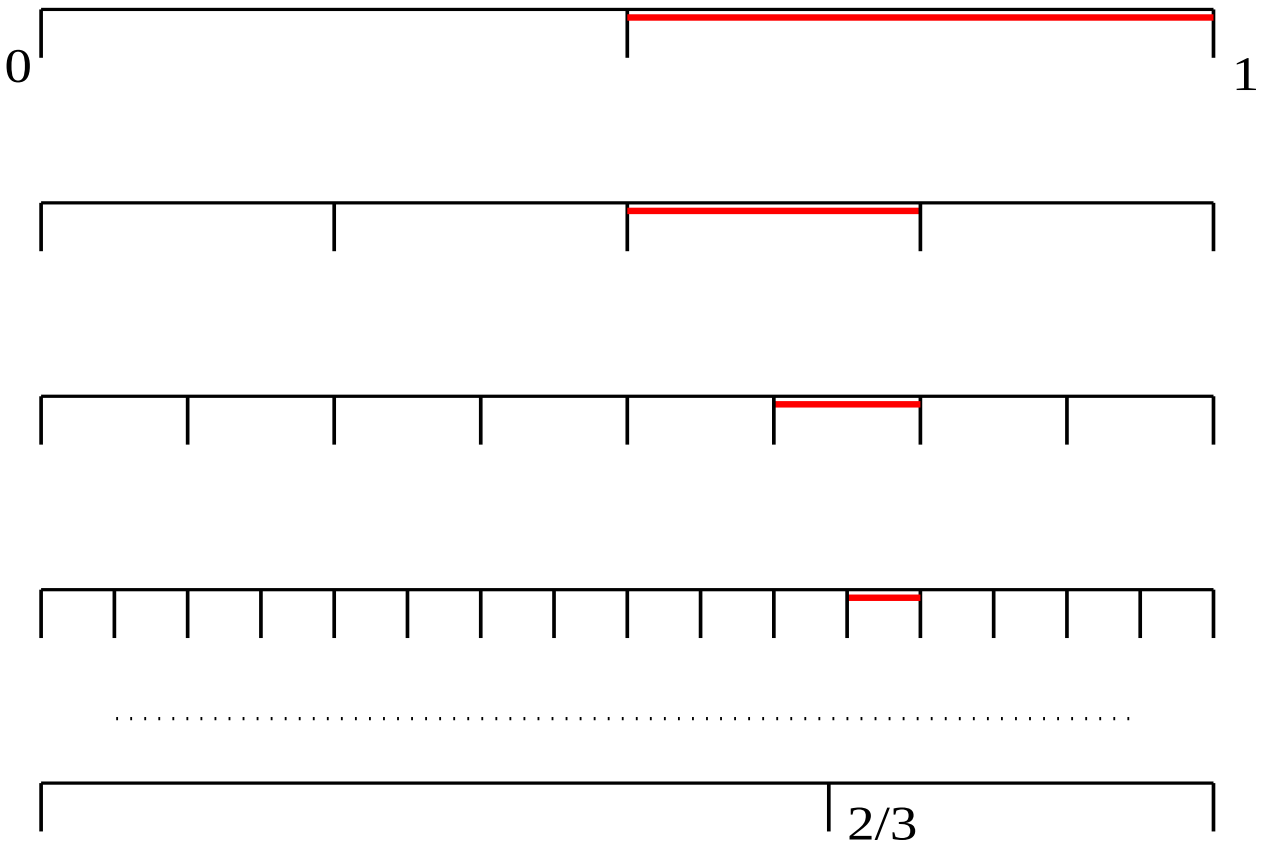
- Given sets $\{p_1, p_2, \ldots\}$ of probabilities, and $\{x_1, x_2, \ldots\}$ of values, there exists a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a random variable X defined on Ω such that $P\{X = x_k\} = p_k$ for all k.
- Take U uniform on [0,1]



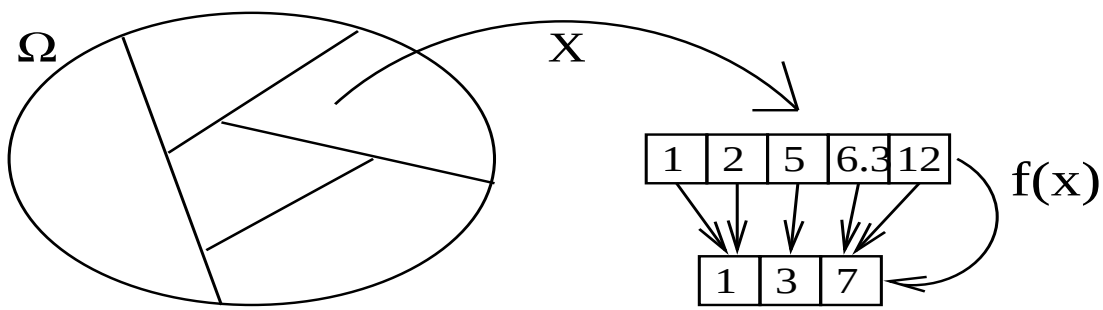
$$
\begin{aligned}
P\{X = x_k\} &= P\{U \in \text{the } k\text{-th interval}\} \\
&= \text{length of the } k\text{-th interval} \\
&= p_k
\end{aligned}
$$

It can be done with one fair coin!

$\cdot 10110011011.....$



Functions of a random variable



$$\boxed{Y \;=\; f(X) \quad \Rightarrow}$$

$$p_Y(y) \equiv P\{Y = y\} \;=\; \sum_{x:f(x)=y} P\{X = x\} \equiv \sum_{x:f(x)=y} p_X(x)$$

$$\boxed{p_Y(y) \;=\; \textstyle\sum_{x:f(x)=y} p_X(x)}$$

$$\boxed{p_Y(y) \;=\; \sum_{x:f(x)=y} p_X(x)}$$

- Take X uniform on $\{-9, -8, \ldots, 8, 9\}$, and $Y = f(X)$.
- Find $p_Y(1)$ and $P\{Y \leq 5\}$.

f	$p_Y(1)$	$P\{Y \leq 5\}$
$2X$	$p_X(1/2) = 0$	$P\{X \leq 5/2\} = 12/19$
$3X - 5$	$p_X(2) = 1/19$	$P\{X \leq 10/3\} = 13/19$
X^2	$p_X(1) + p_X(-1) = 2/19$	$P\{-\sqrt{5} \leq X \leq \sqrt{5}\} = 5/19$
$\lvert X - 2\rvert$	$p_X(3) + p_X(1) = 2/19$	$P\{-3 \leq X \leq 7\} = 11/19$

$$\boxed{p_Y(y) \;=\; \sum_{x:f(x)=y} p_X(x)}$$

- X uniform on $\{-9, -8, \ldots, 8, 9\}$, and $Y = f(X)$.

$$\Rightarrow \qquad P\{X = x\} \;=\; p_X(x) = \begin{cases} 1/19 & \text{if } x \in \{-9, -8, \ldots, 9\}\ , \\ 0 & \text{otherwise.} \end{cases}$$

- Find $p_Y(y)$

f	Range	$p_Y(y)$
$2X$	$\{-18, -16, \ldots, 18\}$	$p_X(y/2)$
$3X - 5$	$\{-32, -29, \ldots, 24\}$	$p_X((y+5)/3)$
X^2	$\{0, 1, 4, \ldots, 81\}$	$2p_X(\sqrt{y}), y \neq 0, \quad p_X(0), y = 0$
$\lvert X - 2\rvert$	$\{0, 1, 2, \ldots, 11\}$	$p_X(y+2) + p_X(2-y)$

Random functions

- Recall, a random variable (rv) is a mapping from a probability space to a set of (real) values.

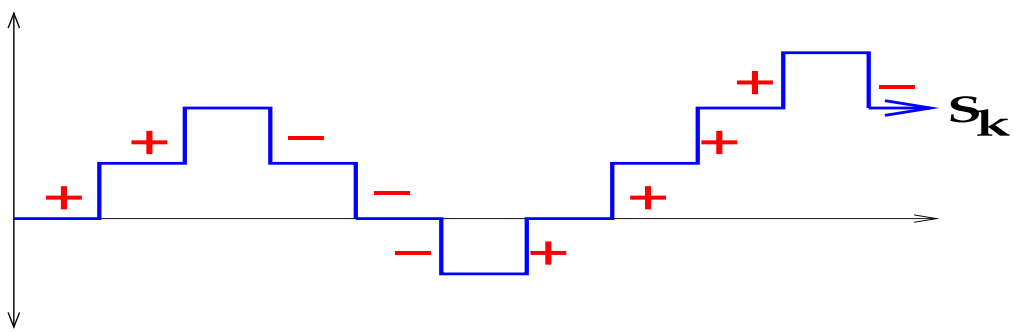
- A random function or stochastic proccess is a mapping from a probability space to a space of functions.

Random walk

- Take $X_1, \ldots, X_n$ independent with $P\{X_j = \pm 1\} = 0.5$
- Define the random sum, or random walk, up to time n by

$$\begin{aligned} S_0 &= 0, \\ S_k &= X_1 + \ldots + X_k, \qquad 1 \le k \le n. \end{aligned}$$

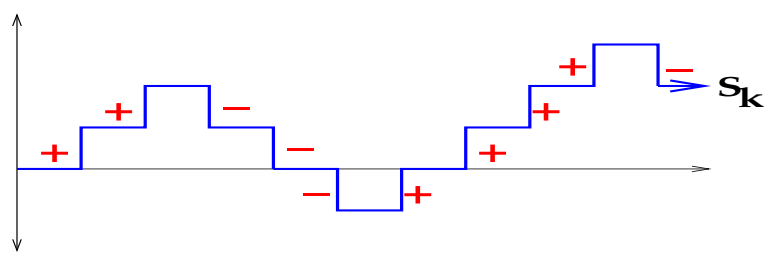
- The probability space is uniform on $\Omega = \{-1, +1\}^n$



A simple random walk calculation

- $P\{S_n = k\} = ???$
- The probability space is uniform on $\Omega = \{-1, +1\}^n$
- Take n and k both even, or both odd.

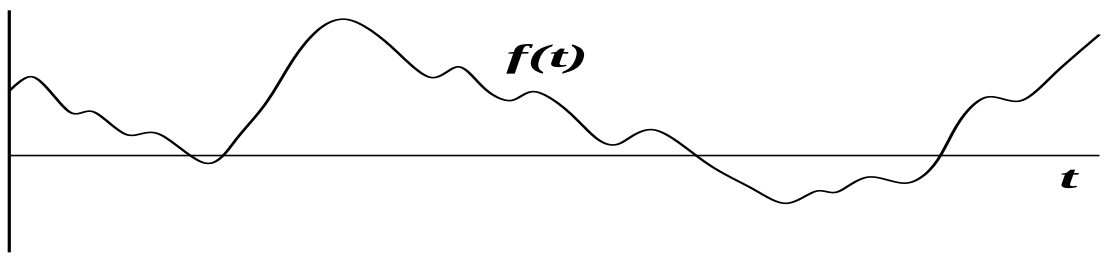
$$
\begin{aligned}
P\{S_n = k\} & \\
&= P\{(n+k)/2 \text{ steps up and } (n-k)/2 \text{ steps down}\} \\
&= \binom{n}{\frac{1}{2}(n-k)} 2^{-n} \qquad\qquad k \in \{-n, -n+2, \ldots, n-2, n\}
\end{aligned}
$$



A continuous process

- Take $X_1, Y_1, \ldots, X_n, Y_n$ independent with $P\{X_j = \pm 1\} = P\{Y_j = \pm 1\} = 0.5 \quad (\Omega = \{-1, +1\}^{2n})$
- Take a set of 'amplitudes' $a_1, b_1, \ldots, a_n, b_n$ and a set of 'frequencies' $\omega_1, \ldots, \omega_n$
- Define the continuous random process $f(t)$ by

$$f(t) = \sum_{k=1}^{n} \Big(X_k a_k \sin(\omega_k t) \; + \; Y_k b_k \cos(\omega_k t) \Big)$$



Joint distributions

If $X_1, \ldots, X_n$ is a collection of integer valued random variables, what can we say about

All of them: $P\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\}$

Some of them: $P\{X_1 = x_1, X_2 = x_2\}$

One at a time: $P\{X_1 = x_1\}$

Conditioning: $P\{X_2 = x_2, \ldots, X_n = x_n | X_1 = x_1\}$

More conditioning: $P\{X_3 = x_3, \ldots, X_n = x_n | X_1 = x_1, X_2 = x_2\}$