Lecture 24

Generating functions

More on normal distributions

Central limit theorem

Continuity theorem

Comments on the CLT

Generating functions

Probability gf $\quad g_X(s) \;=\; E\left\{s^X\right\} \;=\; \sum_x s^x \; p_X(x)$

Moment gf $\quad M_X(t) \;=\; E\left\{e^{tX}\right\} \;=\; \begin{cases} \sum_x e^{tx} \; p_X(x) \\ \int_x e^{tx} \; f_X(x) \; dx \end{cases}$

Characteristic fn $\quad \varphi_X(\theta) \;=\; E\left\{e^{i\theta X}\right\} \;=\; \begin{cases} \sum_x e^{i\theta x} \; p_X(x) \\ \int_x e^{i\theta x} \; f_X(x) \; dx \end{cases}$

where $i = \sqrt{-1}$.

Also, when M_X exists,

$$\varphi_X(\theta) \;=\; M_X(i\theta)$$

Properties of GF's

- Derivatives of MGF's (CF's) functions evaluated at $t = 0$ ($\theta = 0$) give moments. e.g

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = E\{X\}, \qquad \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = E\{X^2\}$$

- $X_1, \ldots, X_n$ independent, with MGF's (or CF's) $M_{X_1}(t), \ldots, M_{X_n}(t)$ $(\varphi_{X_1}, \ldots, \varphi_{X_n})$

$$\begin{aligned} M_{X_1+\cdots+X_n}(t) &= M_{X_1}(t) \cdots M_{X_n}(t) \\ \varphi_{X_1+\cdots+X_n}(\theta) &= \varphi_{X_1}(\theta) \cdots \varphi_{X_n}(\theta) \end{aligned}$$

- MGF's do not always exist, characteristic functions do.

- Both MGF's (when they exist) and CF's determine distributions.

More on normal distributions

$$X \sim N(\mu, \sigma^2) \quad \Rightarrow \quad f(x) \;=\; \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

Therefore,

$$M_X(t) \;=\; E\{e^{tX}\} \;=\; \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{tx}\; e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\; dx$$

Note

$$tx - \frac{(x-\mu)^2}{2\sigma^2} \;=\; \mu t \;+\; \frac{1}{2}\sigma^2 t^2 \;-\; \frac{(x-(\mu-\sigma^2 t))^2}{2\sigma^2}$$

So that

$$M_X(t) \;=\; e^{\mu t + \frac{1}{2}\sigma^2 t^2}\; \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-(x-(\mu-\sigma^2 t))^2/2\sigma^2}\; dx \;=\; e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$\boxed{M_X(t) \;=\; e^{\mu t + \frac{1}{2}\sigma^2 t^2}}$$

Standardising normal distributions

- $X \sim N(\mu, \sigma^2) \iff Z = X^* = \frac{X-\mu}{\sigma} \sim N(0,1)$

PROOF: $(\Rightarrow)$

$$X \sim N(\mu, \sigma^2) \iff M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$\begin{aligned} M_{X^*}(t) &= E\left\{e^{t(X-\mu)/\sigma}\right\} \\ &= e^{-t\mu/\sigma}\, E\left\{e^{\left(\frac{t}{\sigma}\right)X}\right\} \\ &= e^{-t\mu/\sigma}\, e^{\mu\frac{t}{\sigma} + \frac{1}{2}\sigma^2\left(\frac{t}{\sigma}\right)^2} \\ &= e^{\frac{1}{2}t^2} \end{aligned}$$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Sums of normal random variables

- $X \sim N(\mu_X, \sigma_X^2)$, independent of $Y \sim N(\mu_Y, \sigma_Y^2)$,

$$\Rightarrow \qquad (X+Y) \sim N\left(\mu_X + \mu_Y,\ \sigma_X^2 + \sigma_Y^2\right)$$

PROOF:

$$M_X(t) \ = \ e^{\mu_X t + \frac{1}{2}\sigma_X^2 t^2}, \qquad M_Y(t) \ = \ e^{\mu_Y t + \frac{1}{2}\sigma_Y^2 t^2},$$

Thus

$$\begin{aligned} M_{X+Y}(t) &= M_X(t) \cdot M_Y(t) \\ &= e^{\mu_X t + \frac{1}{2}\sigma_X^2 t^2} \cdot e^{\mu_Y t + \frac{1}{2}\sigma_Y^2 t^2} \\ &= e^{(\mu_X + \mu_Y)t + \frac{1}{2}(\sigma_X^2 + \sigma_Y^2)t^2} \end{aligned}$$

- $X_k \sim N(\mu_k, \sigma_k^2),\ k = 1, \ldots, n$, independent,
  $\Rightarrow\ (X_1 + \cdots + X_n) \sim N(\mu_1 + \ldots + \mu_n,\ \sigma_1^2 + \cdots + \sigma_n^2)$

Central limit theorem

- $X_1, X_2, \ldots, X_n$ independent random variables with same distribution. $E\{X_k\} = \mu,\ Var(X_k) = \sigma^2$.
- $S_n = X_1 + \cdots + X_n$

$$P\left\{a \ < \ \frac{S_n - n\mu}{\sqrt{n}\sigma} \ \leq b\right\} \ \approx \ \Phi(b) - \Phi(a)$$

where

$$\Phi(x) \ = \ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\, dx$$

"Continuity Theorem"

- $Z_1, Z_2, \ldots$ random variables with
  moment generating functions $M_{Z_1}, M_{Z_2}, \ldots$
  or characteristic functions $\varphi_{Z_1}, \varphi_{Z_2}, \ldots$

- Z a random variable with MGF $M_Z(t)$ or CF $\varphi_Z(\theta)$.

$$(1) \qquad M_{Z_n}(t) \to M_Z(t) \quad \Longleftrightarrow \quad P\{Z_n \le z\} \to P\{Z \le z\}$$

$$(2) \qquad \varphi_{Z_n}(\theta) \to \varphi_Z(\theta), \ \ \forall\, \theta \quad \Longleftrightarrow \quad P\{Z_n \le z\} \to P\{Z \le z\}$$

for all points z at which the function $P\{Z \le z\}$ is continuous.

- (**??**) must hold for all t for which M_Z is defined.

- (**??**) must hold for all θ.

A proof of the CLT

1: Reduction to $\mu = 0$ and $\sigma^2 = 1$.

$$Z_n \equiv \frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{\sum_1^n (X_k - \mu)/\sigma}{\sqrt{n}} = \frac{\sum_1^n X_k^*}{\sqrt{n}}$$

2: Taylor expansion for small t.

$$\begin{aligned} M_{X^*}(t) &= M_{X^*}(0) + tM'_{X^*}(0) + \frac{t^2}{2}M''_{X^*}(0) + \frac{t^3}{3!}M'''_{X^*}(a) \\ &= 1 + 0 + \frac{t^2}{2} + \frac{t^3}{3!}M'''_{X^*}(a) \\ &= 1 + \frac{t^2}{2} + \frac{t^3}{3!}M'''_{X^*}(a) \end{aligned}$$

for some $0 < a < t$, since $E\{X^*\} = 0$, $Var(X^*) = E\{(X^*)^2\} = 1$.

3: Bounding $M'''_{X^*}(a)$.

$M'''_{X^*}(a) \;=\; E\{X^3 e^{aX}\} \;\equiv\; C$ will be finite if M_{X^*} exists.
(Using characteristic functions, this can be shown to follow from $E\{|X|^3\} < \infty$.)

4: The proof.

$$
\begin{aligned}
E\left\{e^{t\sum_{k=1}^{n} X_k^*/\sqrt{n}}\right\} &= E\left\{e^{tX_1^*/\sqrt{n}}\right\}\cdots E\left\{e^{tX_n^*/\sqrt{n}}\right\} \\
&= E\left\{e^{\frac{t}{\sqrt{n}}X_1^*}\right\}\cdots E\left\{e^{\frac{t}{\sqrt{n}}X_n^*}\right\} \\
&= \left(1 + \frac{(t/\sqrt{n})^2}{2} + C\frac{(t/\sqrt{n})^3}{3!}\right)^n \\
&= \left(1 + \frac{\frac{1}{2}t^2}{n} + \frac{Ct^3/3!}{n^{3/2}}\right)^n \\
&\to e^{\frac{1}{2}t^2} \qquad \text{as } n \to \infty
\end{aligned}
$$

and so the result follows from the continuity theorem.

For a direct proof of the CLT
in the Bernoulli case,
see Pitman, §2.3

What can/cannot be relaxed

- We really should have used φ_X and not M_X, so that we would not have had to assume that all moments exist.
- $Var(X) < \infty$ is enough. Do not need that either $E\{|X^3|\} < \infty$ or that M_X exists.
- $Var(X) < \infty$ is crucial to the Taylor expansion argument.
- Indepedence can be replaced with "weak dependence".
- Identical distributions can be replaced with "none of the X_k significantly larger than the others", and all variances finite.
- $Var(X) = \infty$ is doable, but the answer is qualitatively different: i.e. The limit is no longer normal.