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Lecture notes

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From the lecture note page.

Lecture 1

Four problems in probability

Probability spaces

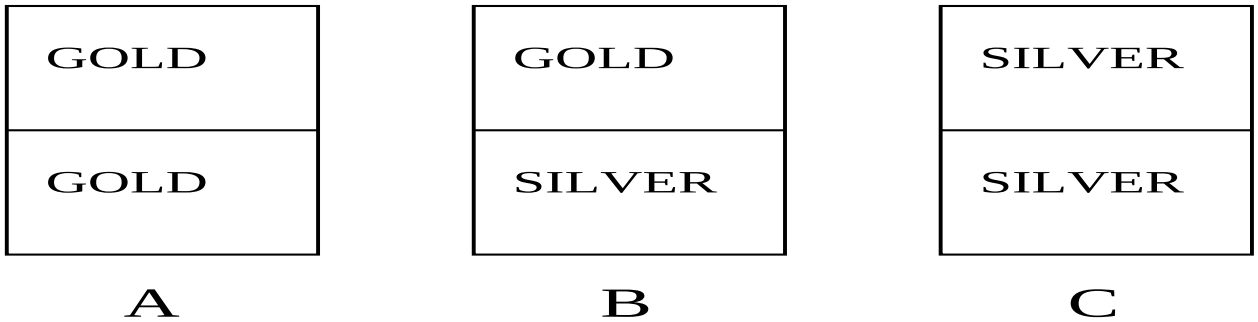
Interpretations of probability

Axioms of probability

Equally likely outcomes

Four problems in probability

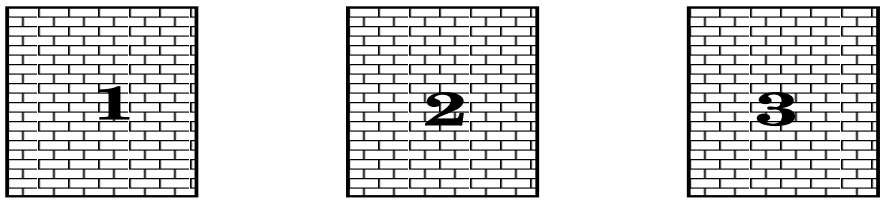
Problem # 1: Coins in drawers



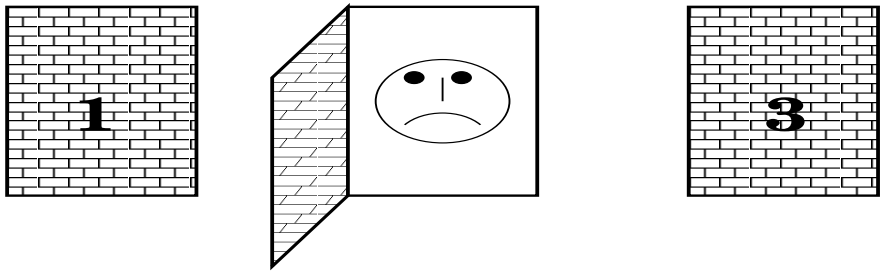
- I choose a drawer at random and find a gold coin.
- What is the probability of a <u>second</u> gold, <u>given</u> that one has already been seen?

Problem # 2: Where is the BMW?

Stage 1:

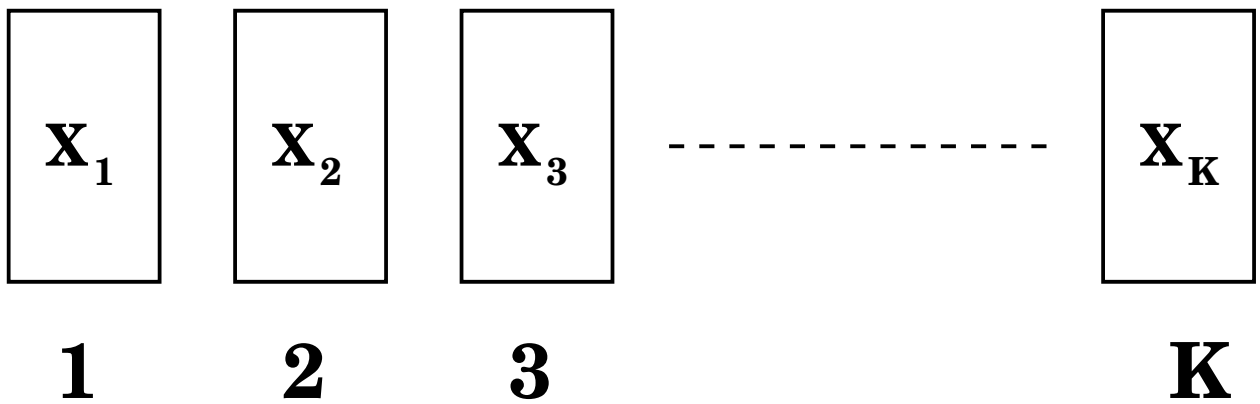


Stage 2:



- I select door 1 at Stage 1.
- What is the probability that the BMW is behind door 3 <u>given</u> that I have just been shown that it is not behind door 2?

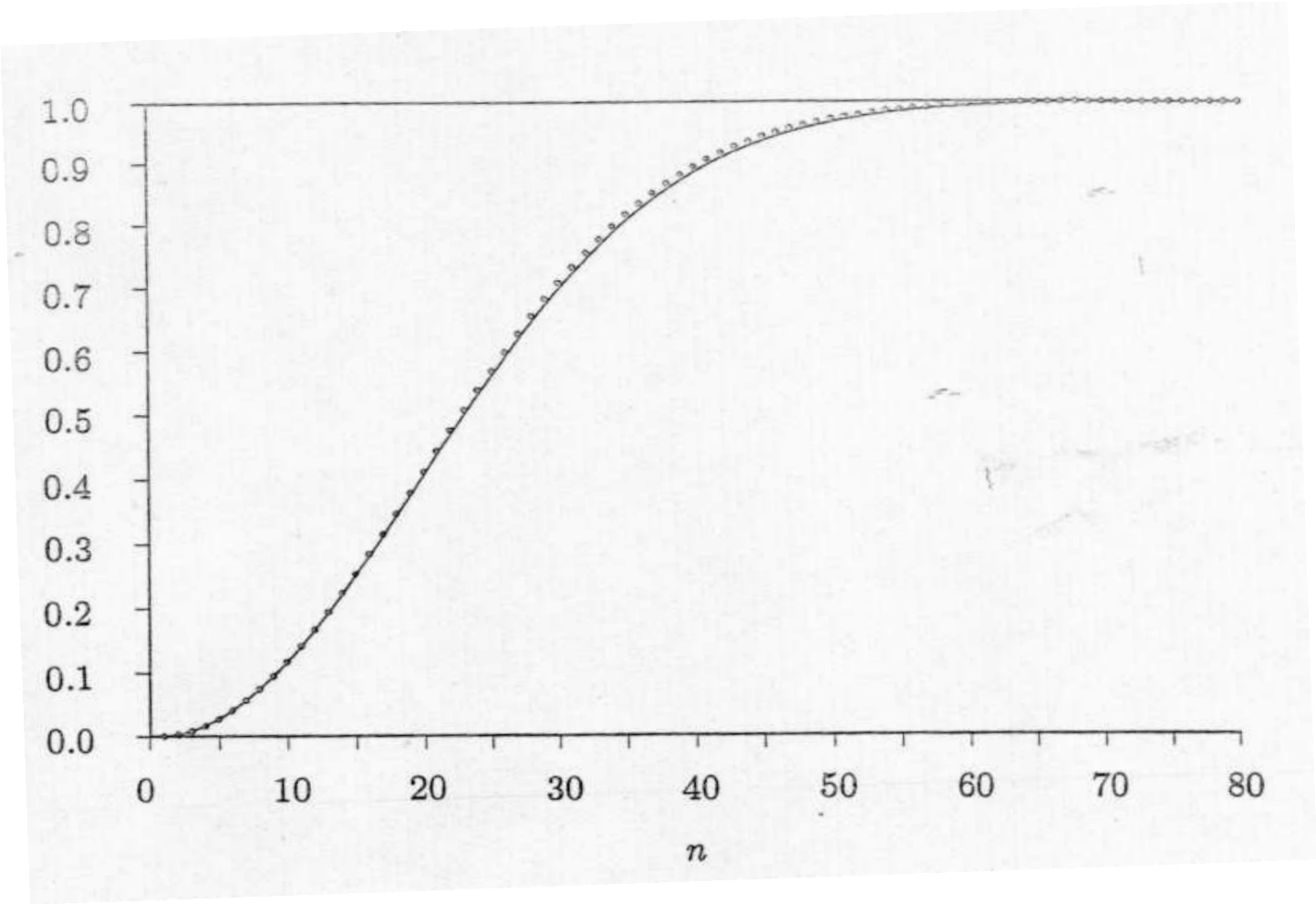
Problem # 3: The birthday problem



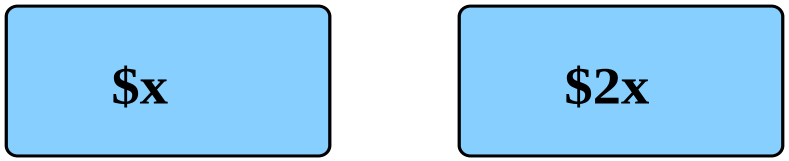
$X_1, X_2, \ldots, X_K$ are chosen <u>at random</u> from the numbers $1, 2, \ldots, N$

- What is the probability that <u>at least 2</u> of the K cards have the same (random) number on them?
- If $N = 365$, and cards are people, this is the "birthday problem".

Birthday probabilities



Problem # 4: The envelope problem



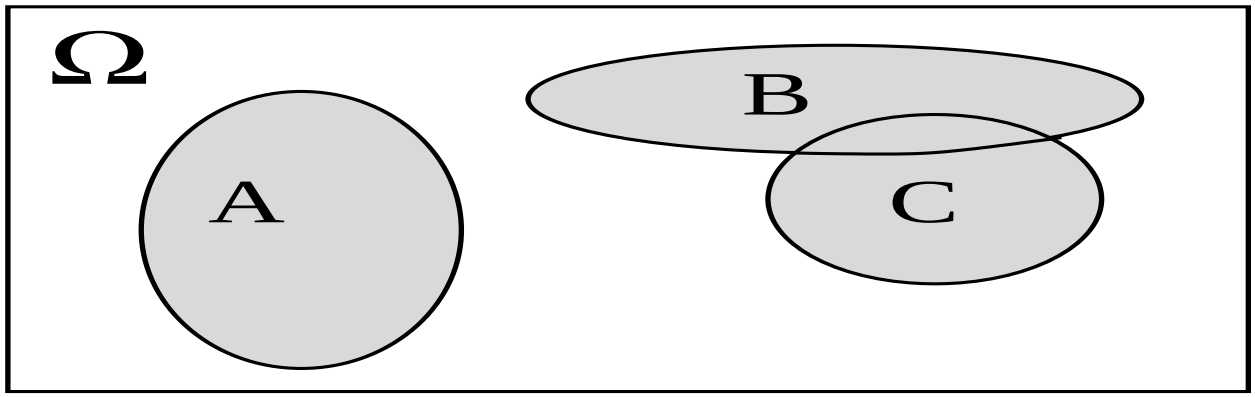
- I choose an envelope with an unknown amount in it. Call it x. I am allowed to exchange envelopes.
- How much do I "expect" there to be in the other envelope?

$$\begin{aligned} \text{Expected amount} &= \frac{1}{2} \times \frac{x}{2} + \frac{1}{2} \times 2x \\ &= 1.25x \end{aligned}$$

- Clearly I should swap, and do so.
- I am allowed to swap again. Should I?
- I still do not know how much I have. Call it x.....

Probability spaces:
$(\Omega, \mathcal{F}, \mathcal{P})$

- $\Omega = \{\omega\}$: Is the *outcome space*, made up of all basic outcomes, or basic events, ω, of a (random) experiment.
- $\mathcal{F}$: Is the *sigma* (σ) *field* of events, $A, B, C, \ldots$, or of sets for which we wish to compute probabilities.
- P: Is the *probability measure*, or *probability distribution*, which assigns probabilities $P(A), P(B), P(C), \ldots$ to events $A, B, C, \ldots$ in $\mathcal{F}$.



EXAMPLES OF PROBABILITY SPACES

	The experiment	
	(Continuous)	(Discrete)
	Choosing a point at random from a square	Tossing a die
Ω	$[0,1]^2 = \{(x,y) : 0 \le x, y \le 1\}$	$\{1,2,3,4,5,6\}$
$\mathcal{F}$	"Measurable" sets $A \subset \Omega$	<u>All</u> subsets of Ω.
	e.g. $A = \{(x,y) : 0 \le x + y \le 1\}$ i.e. The point came from the lower left "corner"	e.g. $A = \{2,4,6\}$ i.e. An even number showed on the die
P	$P\{A\} = \lvert A \rvert \equiv$ area of A	$P\{A\} = (\#A)/6$ $\equiv \dfrac{\text{no. of elements in } A}{6}$

Interpretations of probability

Subjective

- What seems reasonable to me. (California)
- What you are told it is. (Old Moscow)
- What we agree on. (Wall Street)

Frequentist

- What happens in the long run:

i.e. We repeat an experiment n times, in an <u>independent</u> fashion, and watch whether the event A occurs or not. We define

$$P(A) = \lim_{n\to\infty}\left(\frac{\text{Number of times } A \text{ occurs in } n \text{ trials}}{n}\right)$$

The three axioms of probability measures

Non-negativity

For <u>all</u> sets (events) $A \subset \Omega$,

$$P\{A\} \;\geq\; 0.$$

Additivity

If A and B are <u>disjoint</u> ($A \cap B = \emptyset$) then

$$P\{A \cup B\} \;=\; P\{A\} + P\{B\}.$$

Total mass one

$$P\{\Omega\} \;=\; 1.$$

Equally likely outcomes

If Ω is <u>finite</u>, and all <u>basic events</u>, ω, (outcomes) in Ω are <u>equally likely</u>, $(\Rightarrow P\{\omega\} = 1/\#(\Omega))$ then, for all (complex) events $A \subset \mathcal{F}$,

$$P\{A\} = \frac{\#(A)}{\#(\Omega)}$$

Example 1: Tossing a die

- $\Omega = \{\omega_k, k = 1, \ldots, 6\} = \{1, 2, 3, 4, 5, 6\}$
- $A = \{\text{The result is divisible by } 3\} = \{3,6\}$

$$P\{A\} = \frac{\#(\{3, 6\})}{\#(\Omega)} = \frac{2}{6}$$

Example 2: Tossing two dice

- $\Omega = \{\omega_{ij}, i, j = 1, \ldots, 6\} = \{(i, j), i, j = 1, \ldots, 6\}$

$$\Omega = \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{bmatrix}$$

- $P\{\text{Two “ones”}\} = P\{(1,1)\} = 1/36.$
- $P\{\text{A “one” and a “two”}\} = P\{(1,2),(2,1)\} = 2/36.$
- $P\{\text{Sum on two faces} = 7\} = 6/36 = 1/6.$
- $P\{\text{Sum on two faces} \leq 7\} = (6 + 5 + \ldots + 1)/36 = 21/36.$

Example 3: Tossing two dice and seeing only the sum

- $\Omega_1 = \{(i,j), i,j = 1,\ldots,6\}$
- $\Omega_2 = \{2,3,4,5,6,7,8,9,10,11,12\}$

Ω_2 is not a <u>uniform</u> probability space

PROBABILITIES OF BASIC EVENTS IN Ω_2									
Sum	2	3	4	5	6	7	8	...	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	...	$\frac{1}{36}$

$$P\{\text{Sum} = x\} = \frac{6 - |7 - x|}{36}, \qquad x = 2,\ldots,12$$

Two (additional) problems to think about

Who wins first?

- Suppose I play a game against one other player, by tossing coins. I start with \$100, he starts with \$$N$. Every time we see a H, he pays me \$1. Every time we see a T, I pay him \$1. The game ends when one of us loses all his money.

  - What is $P\{\text{I lose my entire bank}\}$?
  - What is $P\{\text{I reach \$200 before losing everything}\}$?
  - What happens as $N \to \infty$?

When does the game end?

- How long does it take until the above game ends?