Lecture 26

Distribution functions

Properties

Examples

A general DF

Expectation from the DF

(Cumulative) distribution functions – (CDF's)

$$F(x) = F_X(x) \equiv P\{X \leq x\}$$
$$= \begin{cases} \int_{-\infty}^{x} f_X(x)\,dx & \text{if } X \text{ has a density } f_X \\ \sum_{x_k \leq x} P\{X = x_k\} & \text{if } X \text{ is discrete} \end{cases}$$

Basic properties

- $F(-\infty) = 0$.
- $F(\infty) = 1$.
- F is <u>non-decreasing</u>. i.e. $x \leq y \quad \Rightarrow F(x) \leq F(y)$.
- F is <u>right continuous</u>. i.e.

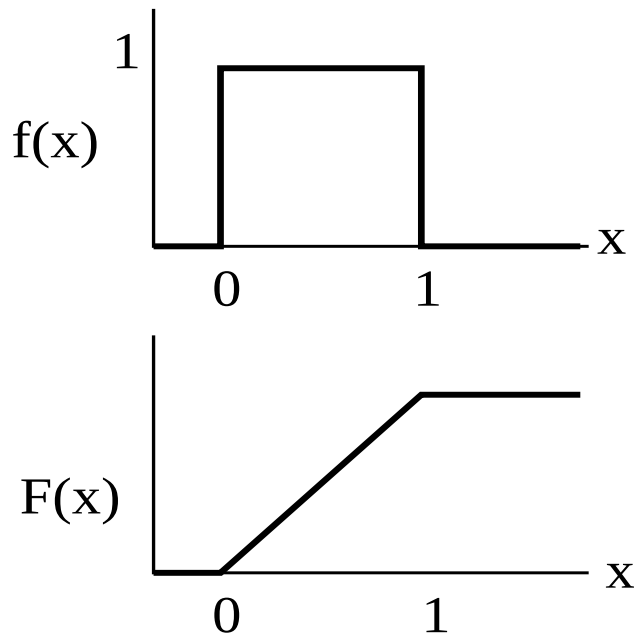
$$\lim_{\delta \to 0, \delta > 0} F(x + \delta) = F(x), \quad \forall x$$

Examples

$X \sim U[0,1]$ (Continuous)

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

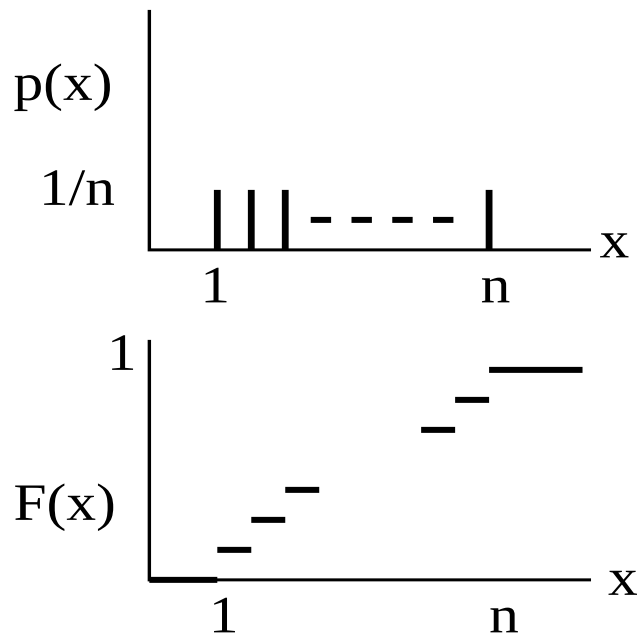
$$F(x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

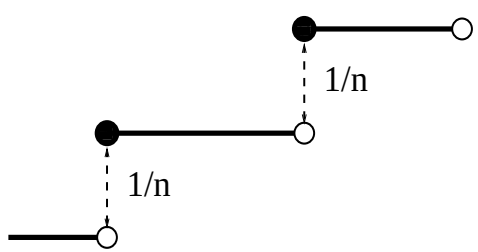


$X \sim U[1, n]$ (Discrete)

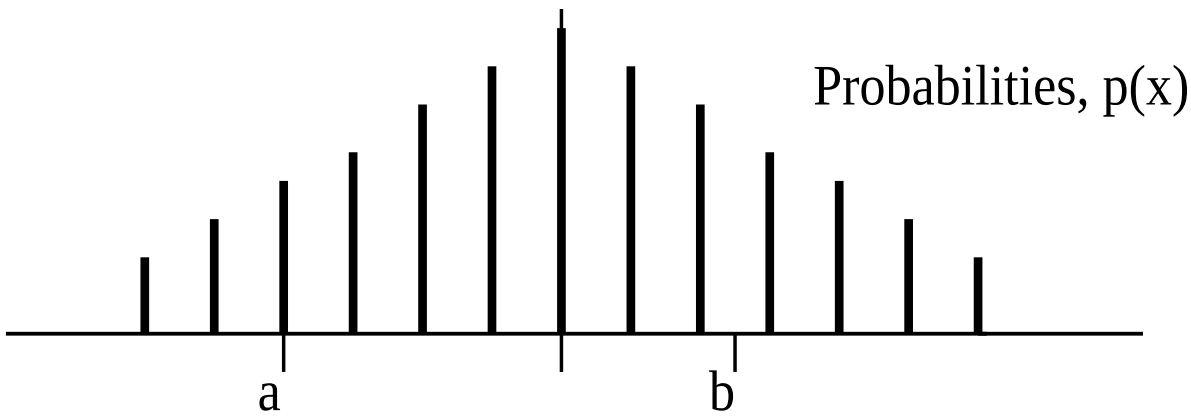
$$p(x) = \begin{cases} 1/n & x \in \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

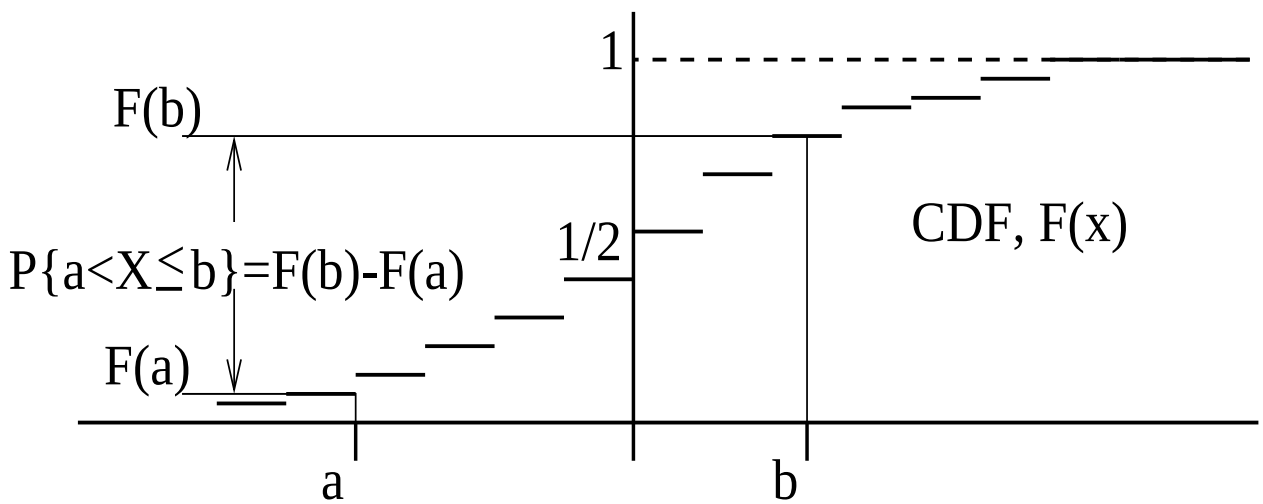
$$F(x) = \begin{cases} 0 & x < 1 \\ k/n & k \leq x < k+1 \\ 1 & x \geq n \end{cases}$$



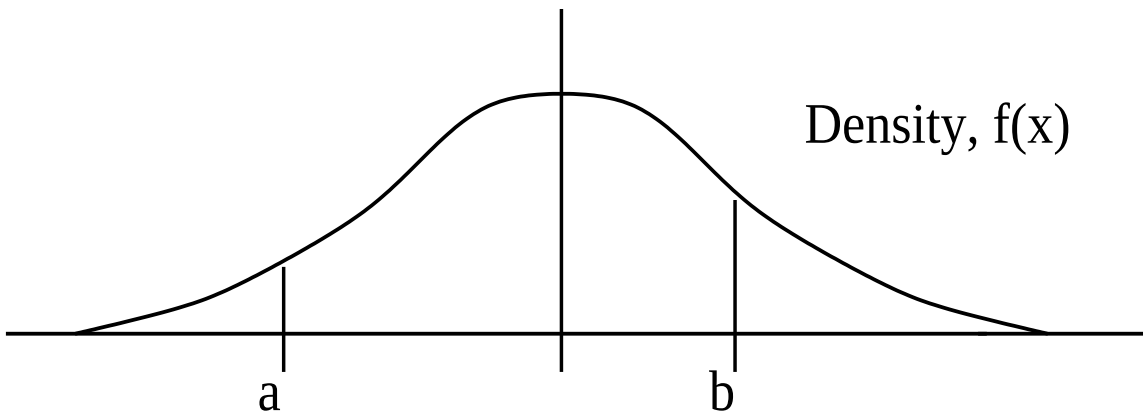


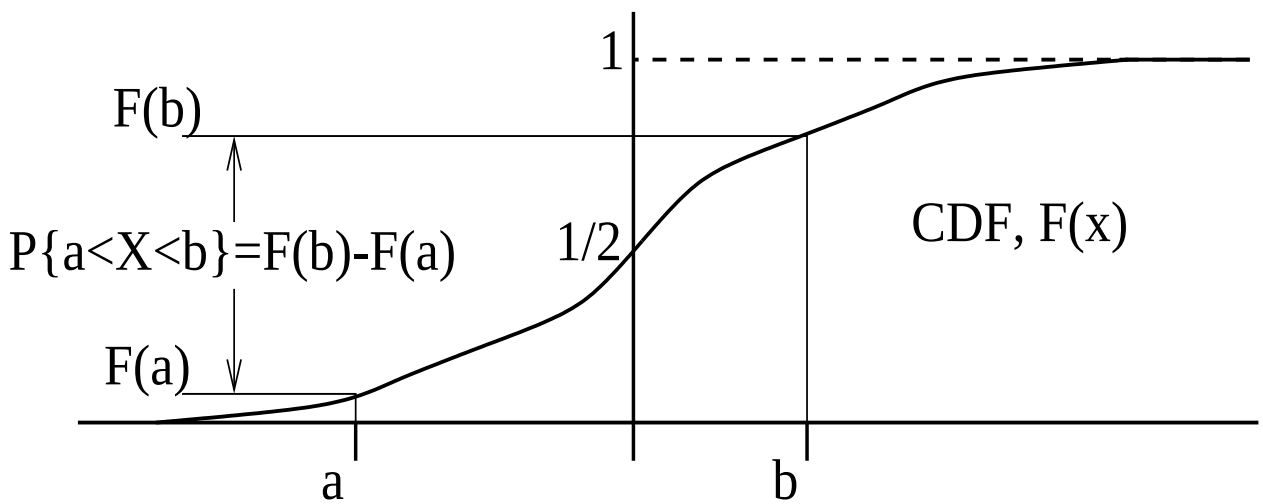
Density and (cumulative) distribution functions





Probability and (cumulative) distribution functions





More properties

- $P\{X > x\} \;=\; 1 - F(x)$, for all X.
- $P\{X \geq x\} \;=\; 1 - F(x)$, for <u>continuous</u> X.
- If X is continuous, then

$$f(x) \;=\; \frac{d}{dx}F(x) \;=\; F'(x)$$

- If X is continuous, then

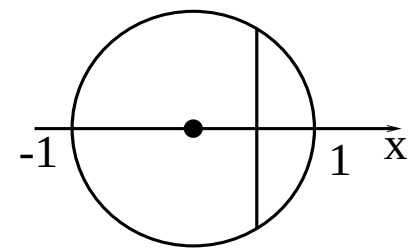
$$\begin{aligned} P\{X \in dx\} &\equiv P\{x < X \leq x + dx\} \\ &= F(x + dx) - F(x) \;= dF(x) \;=\; f(x)\, dx \end{aligned}$$

- If X is discrete, then

$$\begin{aligned} P\{X \in dx\} &\equiv P\{x < X \leq x + dx\} \\ &= F(x + dx) - F(x) \;= dF(x) \;=\; P\{X = x\} \end{aligned}$$

Uniform on a disk

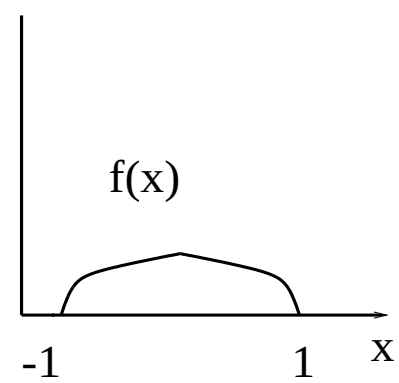
Draw a point at random from the unit disk, and let X denote its first co-ordinate.



$$
\begin{aligned}
F(x) &= P\{X \le x\} \\
&= \left(2\int_{-1}^{x} \sqrt{1-z^2}\, dz\right)/\pi \\
&= \tfrac{1}{2} + \tfrac{1}{\pi}\left[x\sqrt{1-x^2} + \arcsin x\right]
\end{aligned}
$$

which immediately implies that

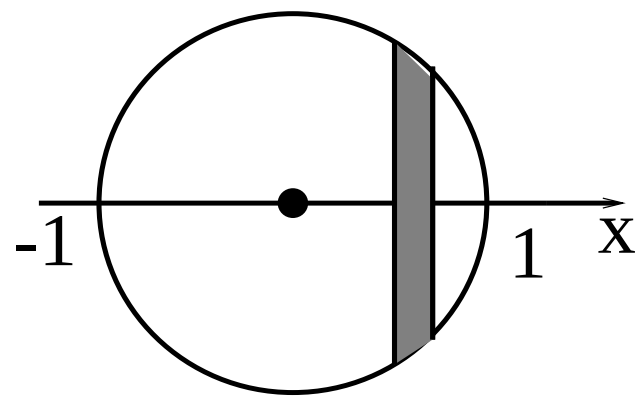
$$
f(x) = \tfrac{2}{\pi}\sqrt{1-x^2}, \qquad -1 \le x \le 1.
$$



Via infinitesimals

Here we find the density $f(x)$ by looking for the <u>infinitesimal</u> probability that $P\{x < X \leq x + dx\} = f(x)\ dx$.

This is immediately given from the diagram as

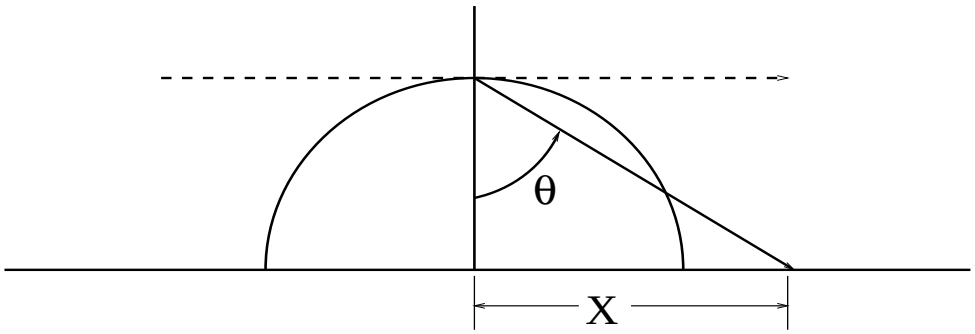


$$f(x)\ dx \quad = \quad \frac{2\sqrt{1-x^2}}{\pi}\ dx$$

so that

$$f(x) \quad = \quad \begin{cases} \frac{2\sqrt{1-x^2}}{\pi} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The Cauchy density – Firing at random



$$F(x) = P\{X \le x\} = P\{\tan\theta \le x\}$$

$$= P\{\theta \le \arctan x\} = \frac{1}{2} + \frac{\arctan x}{\pi}$$

$$\Rightarrow \quad f(x) = F'(x) = \frac{1}{\pi(1+x^2)}, \qquad -\infty < x < \infty$$

NOTE: $E\{X\}$ is undefined, since

$$\int_{-\infty}^{\infty} |x|\ f(x)\ dx = \int_{-\infty}^{\infty} \frac{|x|}{\pi(1+x^2)}\ dx = \infty$$

The most general DF

- Discrete: $F_d(x) \;=\; \sum_{x_k \le x} P\{X = x_k\}$.
- Continuous: $F_c(x) \;=\; \int_{-\infty}^{x} f(x)\, dx$.
- Mixture: There exists a number $\alpha \in [0,1]$ so that

$$F(x) \quad = \quad \alpha F_d(x) \;+\; (1-\alpha) F_c(x)$$

  where F_d is discrete and F_c is continuous.
- Most general: There exist numbers $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, so that

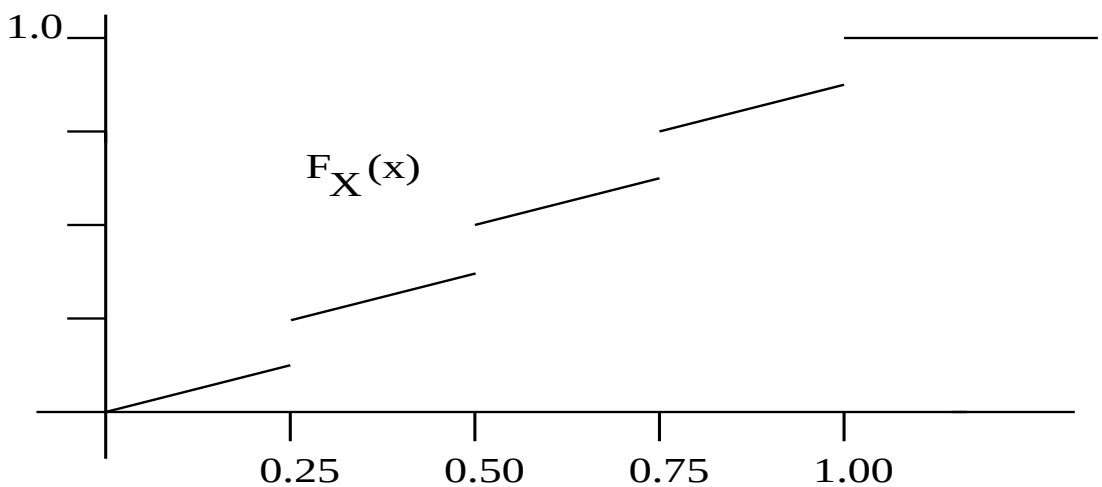
$$F(x) \quad = \quad \alpha F_d(x) \;+\; \beta F_c(x) \;+\; \gamma F_s(x)$$

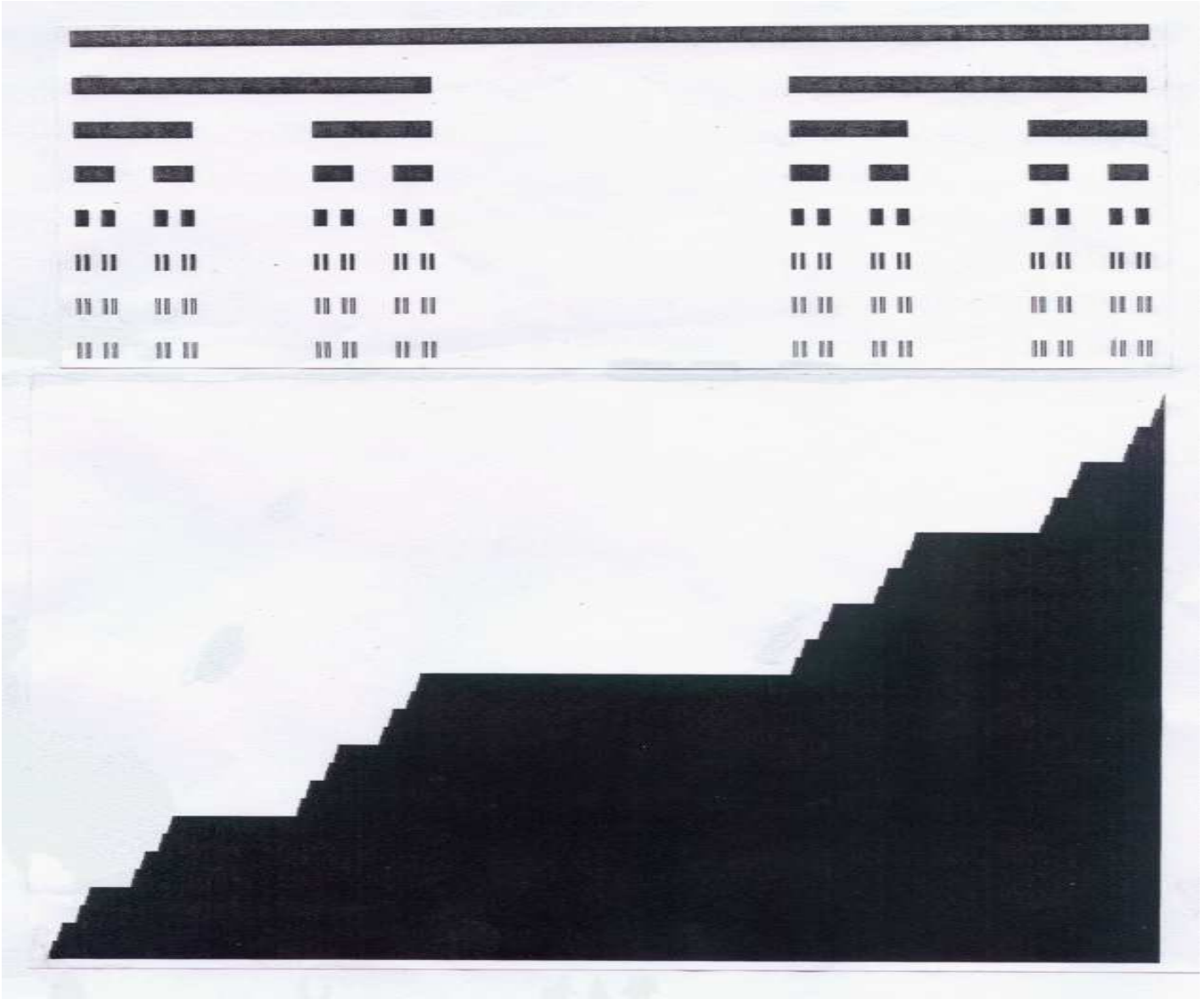
  where F_d is discrete, F_c is continuous, and F_s is "singular".

Example of a mixture DF

- Toss a fair coin. If it shows H, choose one of the numbers (0.25,0,50.0.75,1.0) at random. If it shows T, choose a number at random from [0,1]. Let X denote the result.

- $\alpha = 0.5$, F_d puts probability 1/4 on each of the points (0.25,0,50.0.75,1.0), and $F_c(x) = x$, for $x \in [0, 1]$.





Expectation from the DF

- Recall the formula for <u>discrete</u> random variables taking <u>non-negative, integer</u> values:

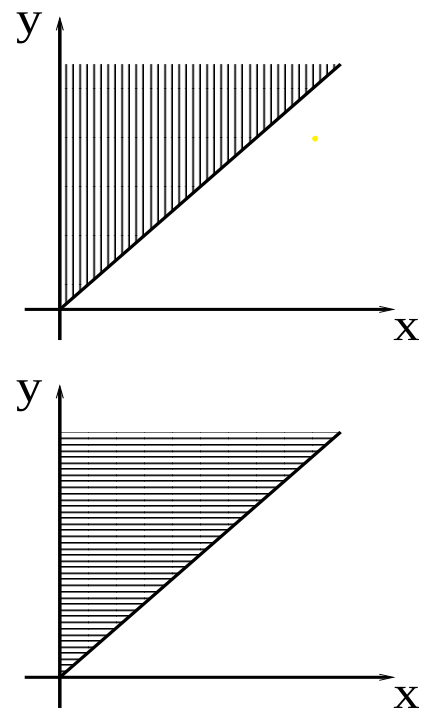
$$
\begin{aligned}
E\{X\} = \sum_{k=1}^{\infty} P\{X \geq k\} &= \sum_{k=1}^{\infty} (1 - F(k-1)) \\
&= \sum_{k=0}^{\infty} (1 - F(k)) \\
&= \int_0^{\infty} (1 - F(x))\ dx
\end{aligned}
$$

- For <u>all</u>, non-negative, random variables we have

$$
E\{X\} = \int_0^{\infty} (1 - F(x))\, dx = \int_0^{\infty} P\{X > x)\ dx
$$

Proof in the density case

$$
\begin{aligned}
&\int_0^\infty P\{X > x)\,dx \\
&\quad = \int_0^\infty \left(\int_x^\infty f(y)\,dy \right)\,dx \\
&\quad = \int_0^\infty \left(\int_0^y f(y)\,dx \right)\,dy \\
&\quad = \int_0^\infty y f(y)\,dy \\
&\quad = \int_0^\infty x f(x)\,dx \\
&\quad = E\{X\}
\end{aligned}
$$



Example

- $X \sim \exp(\lambda)$.
- $F(x) = 1 - e^{-\lambda x} \quad \Rightarrow \quad 1 - F(x) = e^{-\lambda x}$

$$
\begin{aligned}
E\{X\} &= \int_0^\infty (1 - F(x))\, dx \\
&= \int_0^\infty e^{-\lambda x}\, dx \\
&= \frac{1}{\lambda}
\end{aligned}
$$