Lecture 25

Exponential distribution

Lack of memory

Properties

Relation to geometric

Gamma distribution

Wombats and the gamma

Exponential distribution

$$f(t) \;=\; \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- This is a density:

$$\int_{-\infty}^{\infty} f(t)\; dt \;=\; \int_{0}^{\infty} \lambda e^{-\lambda t}\; dt \;=\; \left[-e^{-\lambda t}\right]_0^{\infty} \;=\; 1.$$

- This is a familiar density:

$$P\{T \geq t\} \;=\; \int_{t}^{\infty} \lambda e^{-\lambda t}\; dt \;=\; \left[-e^{-\lambda t}\right]_t^{\infty} \;=\; e^{-\lambda t}$$

which is the probability of no arrival of a Possion process iup to time t.

- $P\{T \geq t\}$ is also know as a survival function.

Lack of memory

- Suppose $T \geq t$. What is the conditional probability that $T \geq t + s$?

$$
\begin{aligned}
P\{T > t + s \mid T > t\} &= \frac{P\{T > t + s \text{ and } T > t\}}{P\{T > t\}} \\
&= P\{T > t + s\} \;/\; P\{T > t\} \\
&= e^{-\lambda(t+s)} \;/\; e^{-\lambda t} \\
&= e^{-\lambda s} \\
&= P\{T > s\}
\end{aligned}
$$

Other properties

- Mean

$$
\begin{aligned}
E\{T\} &= \int_0^\infty t\lambda e^{-\lambda t}\, dt \\
&= \int_0^\infty t \frac{d}{dt}\left(-e^{-\lambda t}\right)\, dt \\
&= \left[-t\, e^{-\lambda t}\right]_0^\infty + \int_0^\infty e^{-\lambda t}\, dt \\
&= 1/\lambda
\end{aligned}
$$

- Variance

$$E\{T^2\} = 2/\lambda^2 \quad \Rightarrow \quad Var(X) = 1/\lambda^2$$

- Standard deviation

$$SD(T) = \sqrt{Var(X)} = 1/\lambda$$

- Moment generating function

$$
\begin{aligned}
M(t) \;=\; E\{e^{tT}\} \;&=\; \int_0^\infty e^{ts}\;\lambda e^{-\lambda s}\;ds \\
&=\; \int_0^\infty \lambda e^{-(\lambda-t)s}\;ds \\
&=\; \frac{\lambda}{\lambda-t}\int_0^\infty (\lambda-t)\;e^{-(\lambda-t)s}\;ds \\
&=\; \frac{\lambda}{\lambda-t} \qquad\qquad (-\infty < t < \lambda)
\end{aligned}
$$

- Variance

$$
E\{T^2\} \;=\; M''(t)\Big|_{t=0} \;=\; \frac{d^2}{dt^2}\left(\frac{\lambda}{\lambda-t}\right)\Big|_{t=0} \;=\; 2/\lambda^2
$$

- Characteristic function

$$
\varphi(\theta) \;=\; E\{e^{i\theta T}\} \;=\; M(i\theta) \;=\; \frac{\lambda}{\lambda-i\theta}
$$

Strontium 90

- Radioactive decay is a Poisson process: λ=?
- "Half life" of material = 28 years
- $\Rightarrow$ "Half life" of <u>nuclei</u> = 28 years

$$1/2 \;=\; P\{\text{Time to decay of } \underline{\text{specific}} \text{ nucleus} > 28\} \;=\; e^{-28\lambda}$$

$$\Rightarrow \lambda \;=\; \frac{\log(1/2)}{-28} \;=\; 0.0248 \text{ per year.}$$

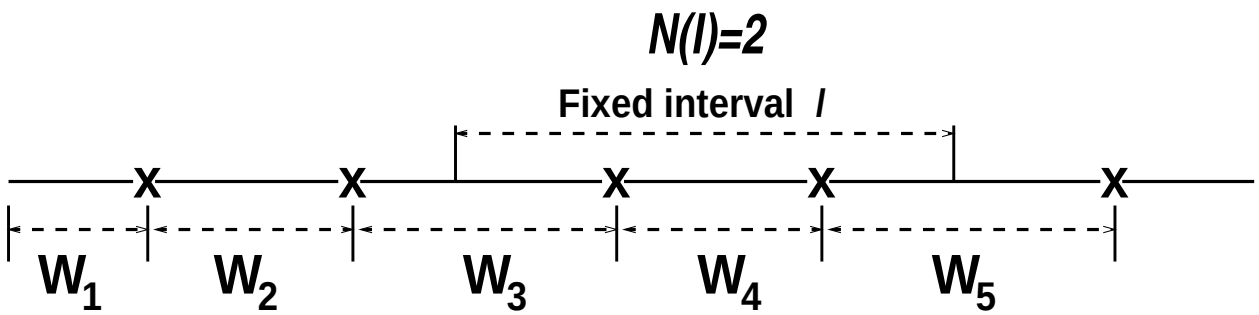
Example: What is the probability that of 5 specific nuclei, at least 3 decay within 2 years?

- $\mu \;=\; 5 \times 2 \times 0.0248 \;=\; 0.248$

$$\begin{aligned} P\{\geq\ 3 \text{ decay}\} &= 1 \;-\; P\{<\ 3 \text{ decay}\} \\ &= 1 - e^{-0.248}[1 + 0.248 + 0.248^2/2] \;=\; .0002 \end{aligned}$$

Equivalent descriptions of a Poisson process

- I: Counts of arrivals. The number of arrivals $N(I)$ in a time interval I of length t is $Poiss(\lambda t)$, an the numbers of arrivals in disjoint intervals are independent.



- II: Times between arrivals. If $W_1, W_2, \ldots$ are waiting times <u>between</u> arrivals, then they are independent and all have a $\exp(\lambda)$ distribution.

Example: Telephone calls

- Telephone calls come into an exchange at an <u>average</u> rate of 3 per minute, as a Poisson process, N. Set $N(s,t) = N(t) - N(s)$.
- Prob'y of no calls in first two minutes.

$$P\{N(0,2) = 0\} \;=\; e^{-2\times 3} \;=\; e^{-6} \;=\; 0.0025$$

- First call takes at least two minutes to arrive.

$$P\{W_1 \geq 2\} \;=\; e^{-3\times 2} \;=\; e^{-6} \;=\; 0.0025.$$

- Prob'y no calls between $t = 0$ and $t = 2$ and at most four calls between $t = 2$ and $t = 3$.

$$\begin{aligned} &P\{N(0,2) = 0\} \cdot P\{N(2,3) \leq 4\} \\ &\qquad = \; e^{-6} \cdot e^{-3} \left(1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{4^2}{4!}\right) \\ &\qquad = \; 0.0020 \end{aligned}$$

Telephone calls (con't)

- Prob'y fifth call takes more than two minutes to arrive

$$
\begin{aligned}
P\{W_1 + W_2 + W_3 + W_4 + W_5 \geq 2\} \\
&= P\{N(0,2) \leq 4\} \\
&= e^{-6}\left(1 + 6 + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!}\right) \\
&= 0.2851
\end{aligned}
$$

- GENERALISATION: Prob'y n-th call takes more than t minutes to arrive

$$
\begin{aligned}
P\{W_1 + \ \ldots \ + \ W_n \geq t\} &= P\{N(0,t) \leq n-1\} \\
&= e^{-\lambda t}\left(1 + (\lambda t) + \frac{(\lambda t)^2}{2!} + \ldots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right)
\end{aligned}
$$

Sums of exponentials

- Define $T = W_1 + \ldots + W_n$, where the W_k are independent exponentials with parameter λ.

Then

$$\begin{aligned} f_T(t) &= \frac{d}{dt}\left[1 - e^{-\lambda t}\left(1 + (\lambda t) + \frac{(\lambda t)^2}{2!} + \ldots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right)\right] \\ &= \lambda\, e^{-\lambda t}\,\frac{(\lambda t)^{n-1}}{(n-1)!} \end{aligned}$$

- Alternatively,

$$\begin{aligned} P\{T \in dt\} &= P\{N(0,t) = n-1,\ N(t, t+dt) = 1\} \\ &\approx \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \times \lambda\, dt \\ \Rightarrow \quad f_T(t) &= \lambda\, e^{-\lambda t}\,\frac{(\lambda t)^{n-1}}{(n-1)!} \end{aligned}$$

Gamma Distribution

$$T \sim \Gamma(n, \lambda) \quad \Longleftrightarrow \quad f_T(t) = \lambda\, e^{-\lambda t}\, \frac{(\lambda t)^{n-1}}{(n-1)!}, \qquad t \geq 0$$

- Moment generating function

$$\begin{aligned} M(t) \;=\; E\{e^{tT}\} \;&=\; \int_0^\infty e^{ts} \lambda\, e^{-\lambda s}\, \frac{(\lambda s)^{n-1}}{(n-1)!}\, ds \\ &=\; \frac{\lambda^n}{(\lambda - t)^n} \int_0^\infty (\lambda - t)\, e^{-(\lambda - t)s}\, \frac{((\lambda - t)s)^{n-1}}{(n-1)!}\, ds \\ &=\; \frac{\lambda^n}{(\lambda - t)^n} \qquad\qquad (0 \leq t < \lambda) \end{aligned}$$

- $M_{\exp(\lambda)} \;=\; \lambda/(\lambda - t)$

$$\Rightarrow \qquad T \sim \Gamma(n, \lambda) \;\Longleftrightarrow\; T = W_1 + \cdots + W_n$$

where $W_1, \ldots, W_n$ are <u>independent</u>, $\exp(\lambda)$ variables.

Properties of $T \sim \Gamma(n, \lambda)$

- Mean: $E\{T\} = E\{W_1 + \cdots + W_n\} = n/\lambda$.
- Variance: $Var(T) = Var(W_1 + \cdots + W_n) = n/\lambda^2$.
- Standard deviation: $SD(T) = \sqrt{n}/\lambda$.
- Right tail probability:

$$
\begin{aligned}
P\{T \geq t\} &= P\{W_1 + \cdots + W_n \geq t\} \\
&= P\{N(0,t) \leq n-1\} \\
&= e^{-\lambda t}\left(1 + (\lambda t) + \frac{(\lambda t)^2}{2!} + \ldots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right)
\end{aligned}
$$

$$
\left(= \int_t^\infty \lambda\, e^{-\lambda t}\, \frac{(\lambda t)^{n-1}}{(n-1)!}\, dt \right)
$$

Gamma distribution for non-integral n

$$T \sim \Gamma(n, \lambda) \quad \Longleftrightarrow \quad f_T(t) = \lambda\, e^{-\lambda t}\, \frac{(\lambda t)^{n-1}}{\Gamma(n)}, \qquad t \geq 0$$

where

$$\Gamma(n) \;\equiv\; \int_0^\infty e^{-t} t^{n-1}\, dt \;\equiv\; \int_0^\infty \lambda e^{-\lambda t} (\lambda t)^{n-1}\, dt$$

- Now n is <u>any</u> non-negative number.
- Moment generating functions remains <u>unchanged</u>.
- $\Rightarrow$ mean, variance, etc. are unchanged.

BUT...

$$T \sim \Gamma(n, \lambda) \quad \Longleftrightarrow \quad f_T(t) = \lambda\, e^{-\lambda t}\, \frac{(\lambda t)^{n-1}}{\Gamma(n)}, \qquad t \geq 0$$

BUT...

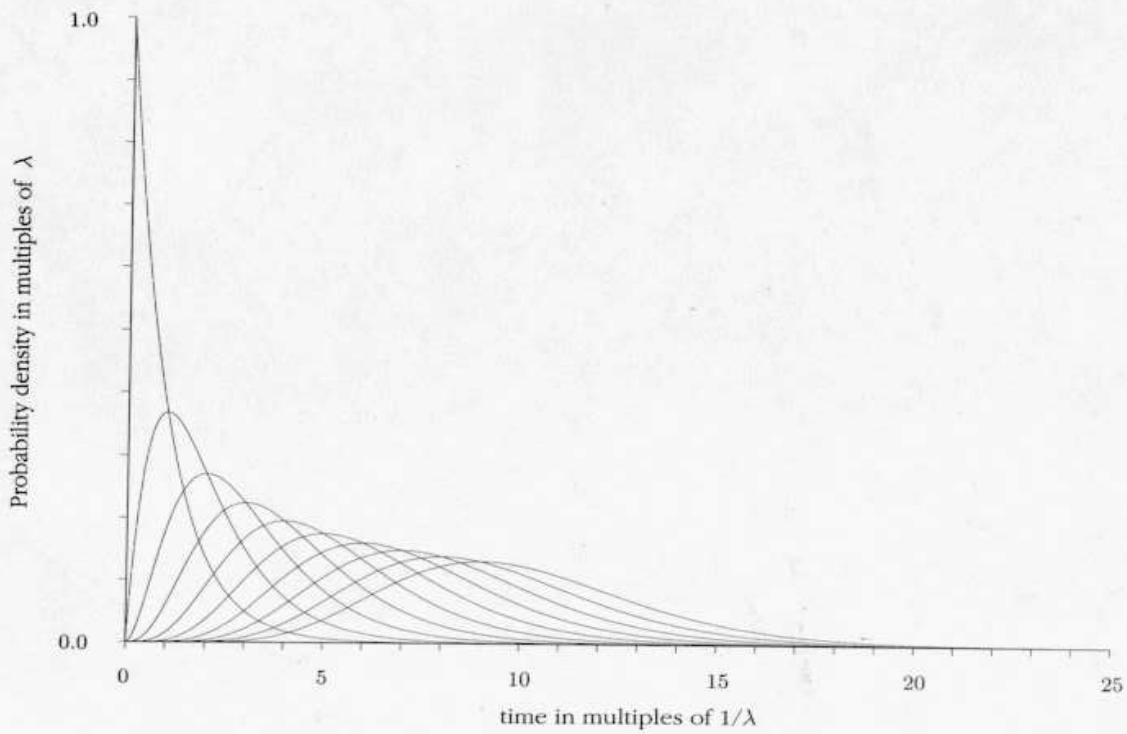
- Representation as a sum of exponentials does not make sense if n is not an integer.
- $\Rightarrow$ right tail probability formula is suspect (and wrong).
- The <u>Gamma function</u>, $\Gamma(n)$ cannot, in general, be evaluated. All we know is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and

$$\Gamma(n) = \begin{cases} (n-1)! & \text{if } n \text{ is an integer.} \\ (n-1)\Gamma(n-1) & \text{always (for } n > 0). \end{cases}$$

(PROOF: Integration by parts and induction on n.)

Gamma densities for integer n (r)

FIGURE 2. Gamma density of the rth arrival for $r = 1$ to 10. Note how the distributions shift to the right and flatten out as r increases, in keeping with the formulae r/λ and $\sqrt{r}/\lambda$ for the mean and SD. Due to the central limit theorem, the gamma (r, λ) distribution becomes asymptotically normal as $r \to \infty$.



Gamma densities for general n (r)

FIGURE 3. Gamma (r, λ) densities for $\lambda = 1$ and r a multiple of $1/4$, $0 < r \leq 1$.

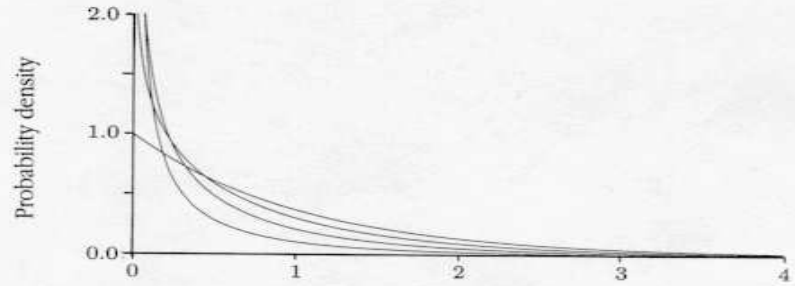


FIGURE 4. Gamma (r, λ) densities for $\lambda = 1$ and r a multiple of $1/4$, $1 \leq r \leq 2$.

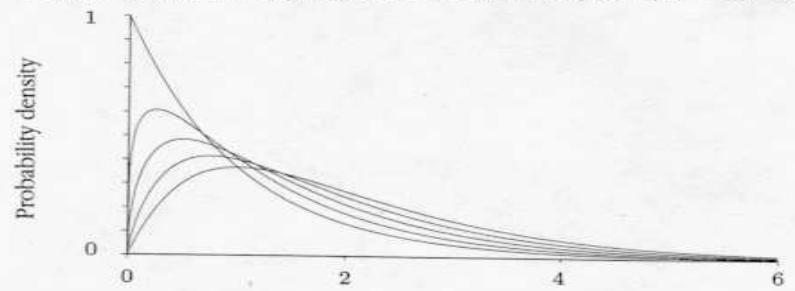
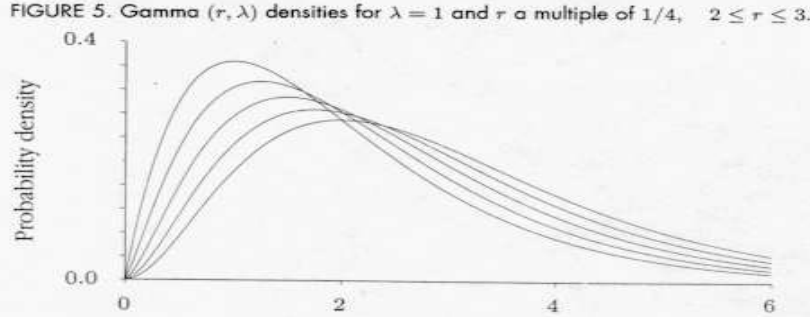



FIGURE 5. Gamma (r, λ) densities for $\lambda = 1$ and r a multiple of $1/4$, $2 \leq r \leq 3$.