Lecture 30

Revision

More $U(D)$ examples

Independence

Examples

Order statistics

Beta distribution

Revision

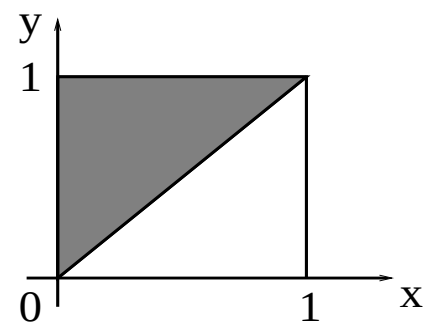
Discrete/Continuous summary table

Handed out in class!

Uniform on a triangle

- Take (X, Y) uniform on the triangle

. $D \equiv \{(x, y) : 0 < x < y < 1\}$.



- To find $f_{XY}(x, y)$, note that

$1: \quad f$ is constant over D

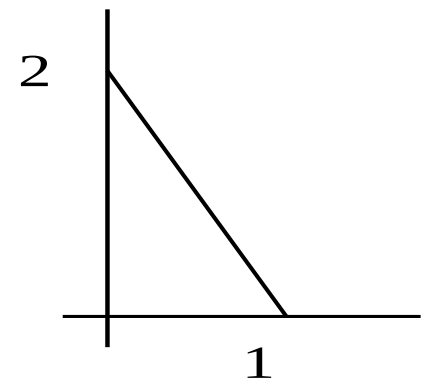
$2: \quad \int_D f(x, y)\, dxdy \;=\; 1$

so that

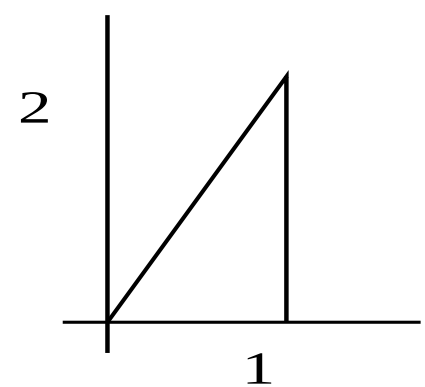
$$f(x, y) \;=\; \begin{cases} 2 & \text{if } (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

- Marginal densities.

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\ dy \\
&= \int_{y=x}^{y=1} 2\ dy \\
&= \begin{cases} 2(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$



$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\ dx \\
&= \int_{x=0}^{x=y} 2\ dx \\
&= \begin{cases} 2y & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$



- Expectations

$$E\{X\} = \int_{-\infty}^{\infty} x\ f_X(x)\ dx = \int_0^1 x\ 2(1-x)\ dx = \frac{1}{3}$$

$$E\{Y\} = \int_{-\infty}^{\infty} y\ f_Y(y)\ dy = \int_0^1 y\ 2y\ dy = \frac{2}{3}$$

- Product expectation

$$\begin{aligned} E\{XY\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ f_{XY}(x,y)\ dxdy \\ &= 2\int_0^1 dy \int_0^y xy\ dx \\ &= 2\int_0^1 \frac{y^3}{2}\ dy = 1/4 \end{aligned}$$

- Are X and Y independent?

Independence of continuous variables

- X and Y independent $\iff$ for all $A, B \subset \Re$,

$$P\{X \in A,\ Y \in B\} \ = \ P\{X \in A\} \cdot P\{Y \in B\}$$

- With $A = dx$ and $B = dy$ this is

$$\begin{aligned} f_{XY}(x,y)\ dxdy &= P\{X \in dx, Y \in dy\} \\ &= P\{X \in dx\} \cdot P\{Y \in dy\} \\ &= f_X(x)dx\ \cdot f_Y(y)dy \end{aligned}$$

$$\boxed{\text{Independence} \iff f_{XY}(x,y) = f_X(x) \cdot f_Y(y)}$$

PROOF:

$\Longrightarrow$ Above.

$\Longleftarrow$ Start with A and B intervals (easy) and keep going.

Independence

$$\iff E(h(X)g(Y)) = E(h(X)) \cdot E(g(Y))$$

for all g and h for which $E(h(X)$ and $E(g(Y))$ are defined.

$\Longrightarrow$ Note that

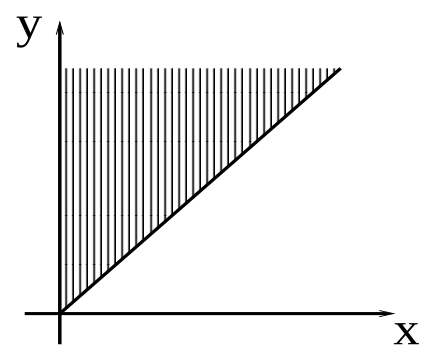
$$\begin{aligned} E(h(X)g(Y)) &= \int\int h(x)g(y)\ f_{XY}(x,y)\ dxdy \\ &= \int\int h(x)g(y)\ f_X(x)f_Y(y)\ dxdy \\ &= \int h(x)\ f_X(x)dx \int g(y)\ f_Y(y)\ dxdy \\ &= E(h(X)) \cdot E(g(Y)) \end{aligned}$$

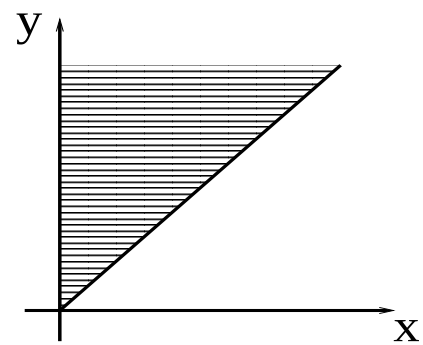
$\Longleftarrow$ Start with $g = 1_A$ and $h = 1_B$ (easy) and keep going.

Independent exponentials

- $X \sim \exp(\lambda),\ Y \sim \exp(\mu)$. Independent
- Compute $P\{X < Y\}$.
- Joint density: (for $x, y > 0$.)

$$f(xy) \;=\; \left(\lambda e^{-\lambda x}\right)\ \left(\mu e^{-\mu y}\right) \;=\; \lambda\mu\ e^{-\lambda x - \mu y}$$





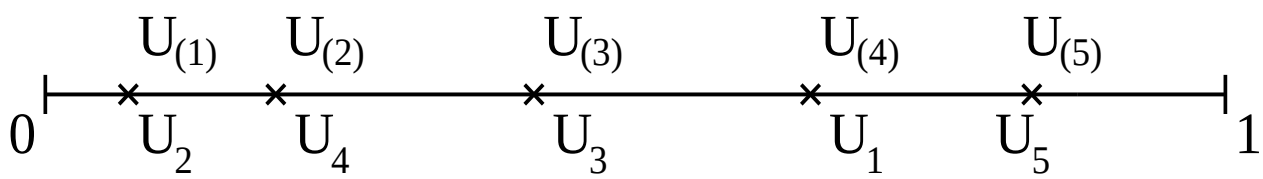
- Therefore:

$$\begin{aligned} P\{X < Y\} &= \int\int_{x<y} \lambda\mu e^{-\lambda x - \mu y}\ dxdy \\ &= \int_{x=0}^{\infty} dx \int_{y=x}^{\infty} \lambda\mu e^{-\lambda x - \mu y}\ dy \\ &= \int_{x=0}^{\infty} \lambda e^{-\lambda x - \mu x}\, dx \\ &= \lambda/(\lambda + \mu) \end{aligned}$$

- Alternatively: $P\{X < Y\} \;=\; \int_{y=0}^{\infty} dy \int_{x=0}^{y} \lambda\mu e^{-\lambda x - \mu y}\ dx$

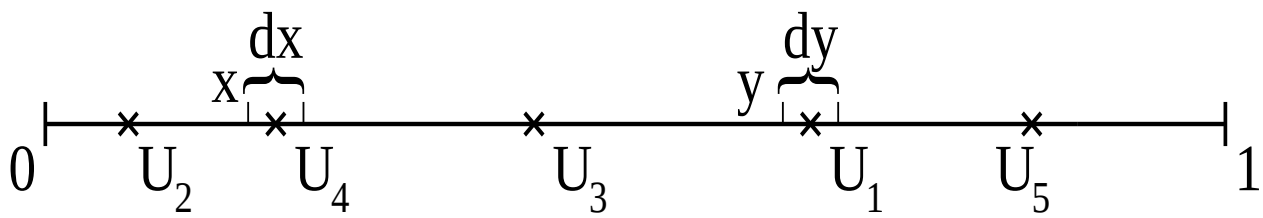
Uniform order statistics

- $U_{(1)} < U_{(2)} < \cdots < U_{(5)}$ are the order statistics of a random sample $U_1, U_2, \ldots, U_5$ from a uniform distribution on [0,1].



- Find the joint density of $X = U_{(2)}$ and $Y = U_{(4)}$.

$$f_{XY}(x,y) = \begin{cases} 5!\; x(y-x)(1-y) & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$
\begin{aligned}
& P\left\{U_{(2)} \in dx, U_{(4)} \in dy\right\} \\
&= P\{\text{one } U_i \text{ in } (0,x), \text{ one in } dx, \text{ one in } (x,y), \\
&\qquad\qquad \text{one in } dy, \text{ one in } (y,1)\} \\
&= 5!\ P\{U_2 \in (0,x),\ U_4 \in dx,\ U_3 \in (x,y),\ U_1 \in dy,\ U_5 \in (y,1)\} \\
&= 5!\ x\ dx\ (y-x)\ dy\ (1-y)
\end{aligned}
$$

- Example: Find $p = P\{U_{(2)} > 1/4,\ U_{(4)} > 1/2\}$

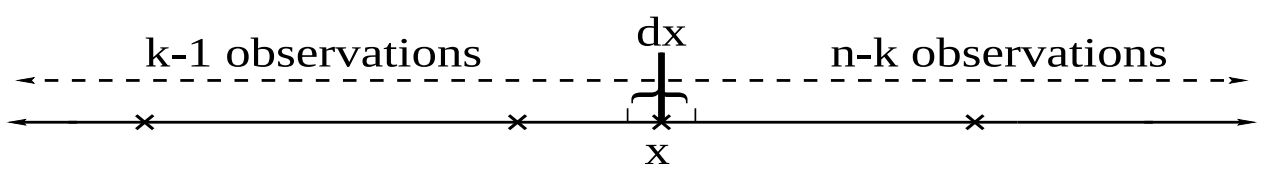
$$
p \ = \ \int_{y=1/2}^{1} \int_{x=1/4}^{y} 5! x(y-x)(1-y)\ dxdy \ = \ \frac{27}{64}
$$

General order statistics

- $X_1, \ldots, X_n$ is a random sample from a distribution with density f and distribution function F.
- $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ are the order statistics.
- Distribution of $X \equiv X_{(k)}$ is given by

$$f_{(k)}(x) \quad = \quad n\ f(x) \binom{n-1}{k-1} (F(x))^{k-1}(1-F(x))^{n-k}$$

$$(-\infty < x < \infty)$$



- We just showed

$$f_{(k)}(x) \quad = \quad n\ f(x) \binom{n-1}{k-1} (F(x))^{k-1}(1-F(x))^{n-k}$$

- Note:

$$f_{(n)}(x) \ = \ f_{\max}(x) \ = \ nf(x)(F(x))^{n-1} \ = \ \frac{d}{dx}\left(F^n(x)\right)$$

$$\begin{aligned} f_{(1)}(x) \ = \ f_{\min}(x) \ &= \ nf(x)(1-F(x))^{n-1} \\ &= \ \frac{d}{dx}\left(1-(1-F(x))^n\right) \end{aligned}$$

- But, f_k is not so simple!

Back to the uniform

- $U_{(1)} < U_{(2)} < \cdots < U_{(n)}$ are the order statistics of a random sample $U_1, U_2, \ldots, U_n$ from a uniform distribution on [0,1].

$$\begin{aligned} f_{(k)}(x) &= n\binom{n-1}{k-1} x^{k-1}(1-x)^{n-k} \\ &= \frac{n!}{(n-k)!(k-1)!} x^{k-1}(1-x)^{n-k} \end{aligned}$$

"Beta" distribution:

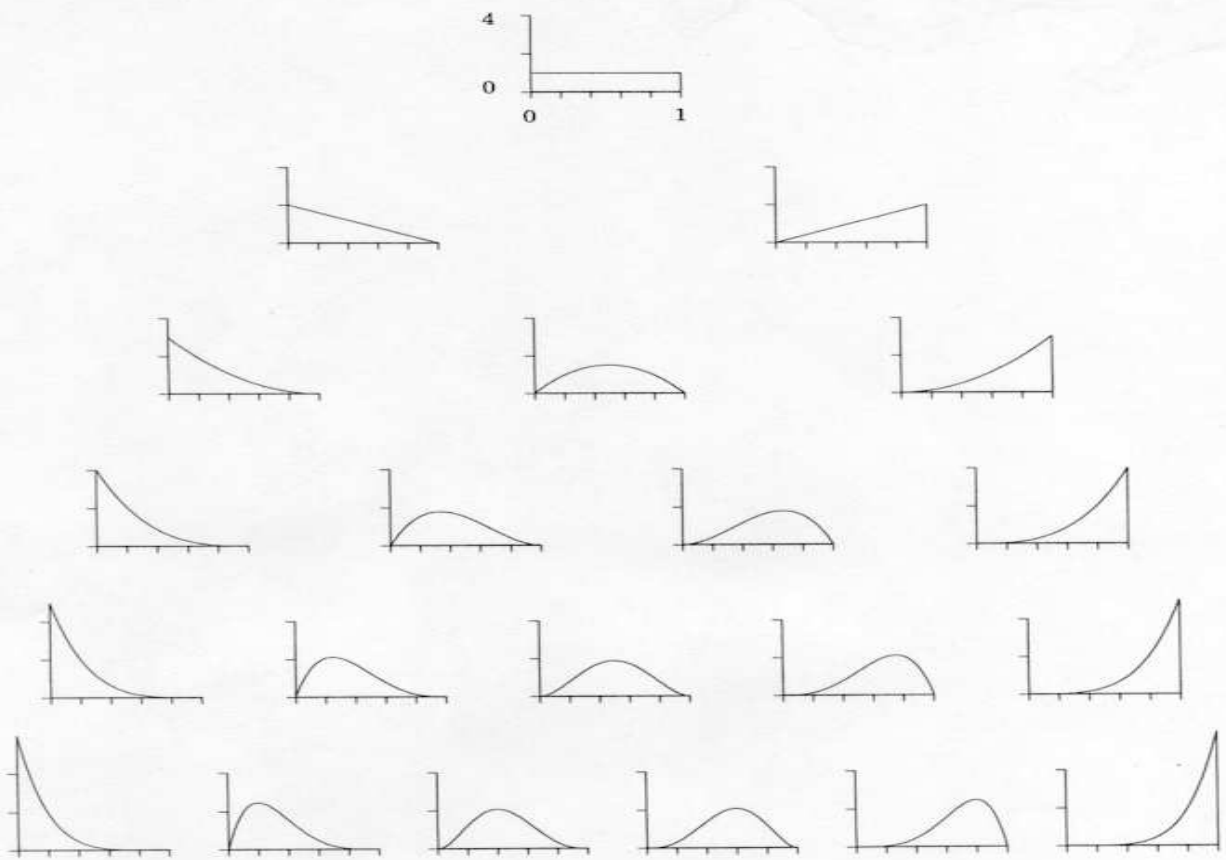
$$f(x) = \frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1} \qquad 0 < x < 1$$

where $B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx$

- $U_{(k)} \sim \text{Beta}(k, n-k+1)$

Some Beta distributions

FIGURE 1. **Densities of order statistics of independent uniform variables.** For $n = 1, 2, \ldots, 6$ and $k = 1, 2, \ldots, n$, the density of the kth order statistic of n independent uniform $(0, 1)$ random variables, which is the beta density with parameters k and $n - k + 1$, is plotted as the kth graph in the nth row of the diagram.



Properties of the Beta distribution

- From computation on the uniform order statistics, or by integration by parts,

$$B(r,s) \;=\; \int_0^1 x^{r-1}(1-x)^{s-1}\,dx \;=\; \frac{(r-1)!\;(s-1)!}{(r+s-1)!} \;=\; \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

- $X \sim \text{Beta}(r,s) \;\Rightarrow$

$$\begin{aligned} E\{X\} &= \frac{1}{B(r,s)}\int x\; x^{r-1}(1-x)^{s-1}\,dx \\ &= \frac{B(r+1,s)}{B(r,s)} \\ &= \frac{r}{r+s} \end{aligned}$$

- Back to uniform order statistics, $E\{X_{(k)}\} = k/(n+1)$.