Lecture 14

Expectation

Definition

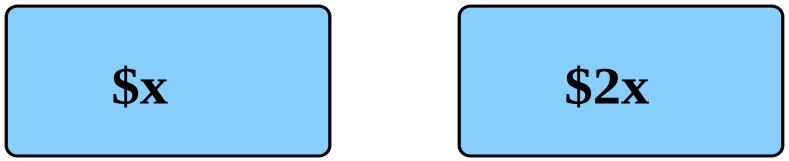
Examples

Interpretations

Measures of centrality

Additivity

Problem # 4: The envelope problem



- I choose an envelope with an unknown amount in it. Call it x. I am allowed to exchange envelopes.
- How much do I "expect" there to be in the other envelope?

$$\begin{aligned} \text{Expected amount} &= \frac{1}{2} \times \frac{x}{2} + \frac{1}{2} \times 2x \\ &= 1.25x \end{aligned}$$

- Clearly I should swap, and do so.
- I am allowed to swap again. Should I?
- I still do not know how much I have. Call it x.....

Mean of a sample

- Sample $x_1, x_2, \ldots, x_n$ has a <u>mean</u>

$$\bar{x}_n \equiv \frac{1}{n}\sum_{k=1}^{n} x_k = \sum_{j=1}^{m} v_j p_j$$

where the v_j are the values taken by the sample and the p_j are the proportions of each value in the sample.

- Example: Sample $= (6, 3, 3)$, $\bar{x}_3 = (6+3+3)/3 = 4$,
  $v_1 = 3,\ v_2 = 6,\ p_1 = 2/3,\ p_2 = 1/3,$

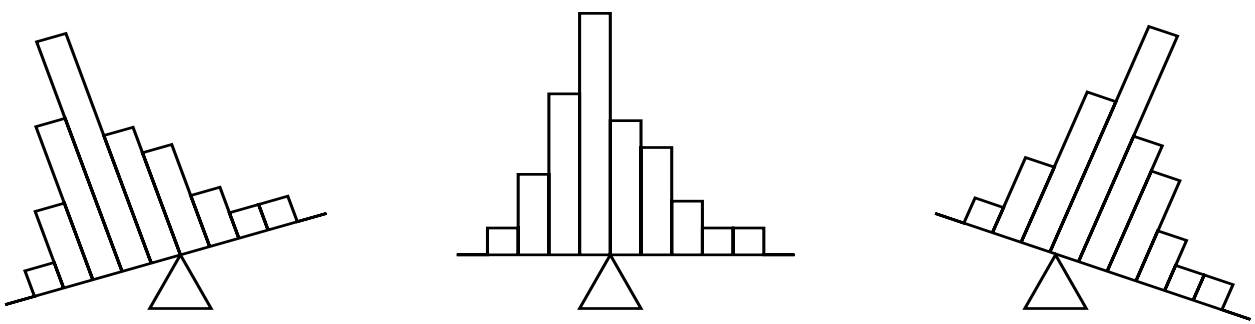
$$\sum_{j=1}^{m} v_j p_j = 3\left(\frac{2}{3}\right) + 6\left(\frac{1}{3}\right) = 4.$$

Expectation

- X is a <u>discrete</u> random variable taking possible values $\{x_1, x_2, \ldots\}$ with probabilities $p_k = P(X = x_k)$.

$$E\{X\} \equiv \sum_k x_k p_k.$$

if $\sum_k |x_k| p_k < \infty$.



Examples

Two possible values X takes only the values a and b with probabilties p_a and p_b.

$$E\{X\} \;=\; ap_a \;+\; bp_b.$$

Indicator variable $X = 1_A$.

$$\begin{aligned} E\{X\} &= 1 \times P\{X=1\} \;+ 0 \times P\{X=0\} \\ &= P\{A\}. \end{aligned}$$

Rolling a die

$$\begin{aligned} E\{X\} &= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) \\ &= (1+2+3+4+5+6)/6 \;=\; 3.5 \end{aligned}$$

Binomial variable

$$
\begin{aligned}
E\{X\} &= \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=0}^{n-1} (k+1) \binom{n}{k+1} p^{k+1} q^{n-(k+1)} \\
&= \sum_{k=0}^{n-1} \frac{(k+1)n!}{(k+1)!(n-(k+1)!} p^{k+1} q^{n-(k+1)} \\
&= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!((n-1)-k)!} p^k q^{(n-1)-k} \\
&= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k} \qquad = np
\end{aligned}
$$

Poisson variable

$$
\begin{aligned}
E\{X\} &= \sum_{k=0}^{\infty} k \, \frac{e^{-\lambda}\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} \frac{k\,\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} \\
&= \lambda \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \\
&= \lambda
\end{aligned}
$$

A sure bet

- I play a "fair bet" game: For a bet of $\$a$, I lose all with probability 1/2 and win $\$2a$ with probability 1/2.

$$E\{\text{Win per game}\} \;=\; a \cdot 0.5 \;+\; (-a) \cdot 0.5 \;=\; 0.$$

- Policy: On game $\#n$ I bet $\$2^{n-1}$ and play until I win.

- If I win on game $\#n$, I win

$$2^n - \left(1 + 2 + 2^2 + \ldots + 2^{n-1}\right) \;=\; 2^n - (2^n - 1) \;=\; 1.$$

- T = Number of games until a win, $\Rightarrow$ T has a geometric distribution with parameter $p = 1/2$.

$$P\{T = t\} \;=\; (1/2)^t$$

$$E\{T\} \;=\; \sum_{t=1}^{\infty} t(1/2)^t \;=\; 2. \qquad \text{(Next lecture)}$$

- W = Profit

$$W \equiv 1 \;\Rightarrow\; E\{W\} = 1$$

- I investment required to play until victory

- If $T = t$, then

$$I \;=\; \left(1 + 2 + 2^2 + \ldots + 2^{t-1}\right) \;=\; 2^t - 1$$

- Expected investment (Saint Petersburg paradox).

$$\begin{aligned} E\{I\} &= \sum_{t=1}^{\infty} \left(2^t - 1\right)\; 2^{-t} \\ &= \sum_{t=1}^{\infty} 2^t\; 2^{-t} \;-\; \sum_{t=1}^{\infty} 2^{-t} \\ &= \infty \;-\; 1 \\ &= \infty \end{aligned}$$

"Cauchy" variable

$$p_X(x) = \frac{C}{1+x^2}, \qquad x = 0, \pm 1, \pm 2, \ldots$$

$$E\{X\} = C \sum_x \frac{x}{1+x^2},$$

- BUT, we must have $\sum_x |x| p_X(x) < \infty$.

$$\begin{aligned}
\sum_x |x| p_X(x) &= C \lim_{T\to\infty} \sum_{x=-T}^{+T} \frac{|x|}{1+x^2} \\
&\geq \frac{C}{2} \lim_{T\to\infty} \sum_{x=1}^{T} \frac{1}{x} \\
&= \infty
\end{aligned}$$

- $\Rightarrow$ expectation "does not exist" or "is infinite":

$$E\{X\} \neq 0!$$

Interpretations of Expectation

1: Long run average

- The "law of large numbers"

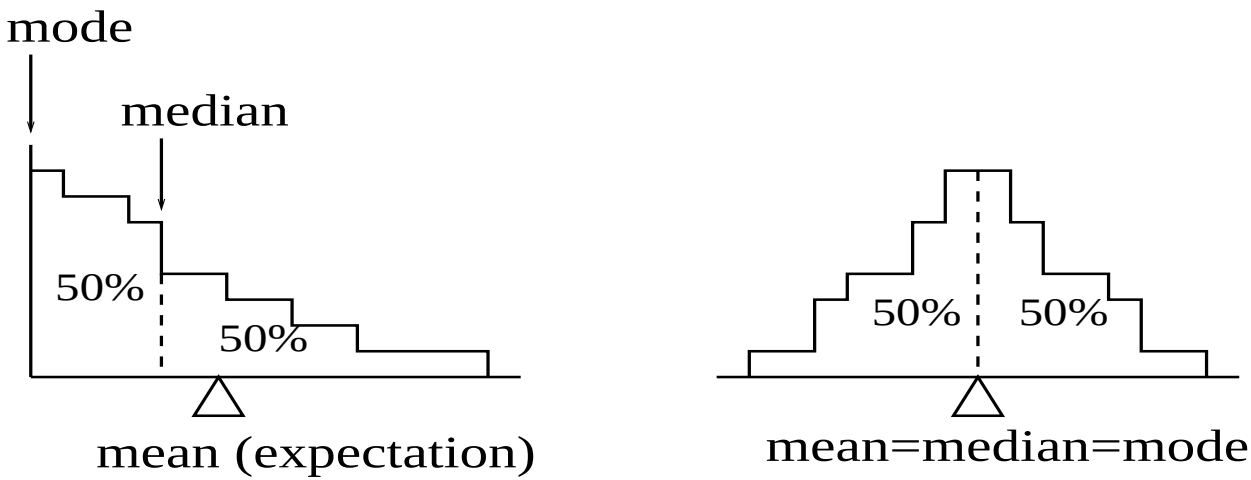
$$\bar{x}_n \;\rightarrow\; E\{X\},$$

as the sample size $n \rightarrow \infty$.

2: Gambling and a "fair price"

- A game has a random "winning" of \$X.
- N games have a random winning of $S_N = X_1 + \ldots + X_N$, where the X_k are independent "copies" of X.
- LLN $\Rightarrow$ the long run gross return per bet will be $E\{X\}$.
- $\Rightarrow$ $E\{X\}$ is a "fair" price to pay to play the game.

Measures of centrality: Mean, median, mode



Probabilities	Median	Mode	$E\{X\}$
$p_1 = .5,\ p_2 = .499999,\ p_3 = 10^{-6}$	1	1	1.500001
$p_1 = .5,\ p_2 = .499999,\ p_{10^7} = 10^{-6}$	1	1	11.499998

Addition rule

$$E\{X+Y\} = E\{X\}+E\{Y\}$$

- Makes sense.
- Requires no conditions beyond existence of the expectations.
- Needs a proof.
- Immediate extension:

$$E\{X_1 + X_2 + \ldots + X_n\} \;=\; E\{X_1\} + E\{X_2\} + \ldots + E\{X_n\}.$$

Example: Tossing two dice

- X = number on die 1.
- Y = number on die 2.

$$E\{X\} = E\{Y\} = 3.5 \Rightarrow E\{X+Y\} = 3.5 + 3.5 = 7.$$

Sum	2	3	4	5	6	7	8	...	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	...	$\frac{1}{36}$

$$p(x) = \frac{6-|7-x|}{36} \Rightarrow E\{X\} = \frac{1}{36}\sum_{x=2}^{12} x\,(6-|7-x|)$$
$$= 7$$