Lecture 31

Discrete conditional distributions

Conditional expectation given an event

$E(Y|X)$

Expectation of a random sum

Sharkey's casino

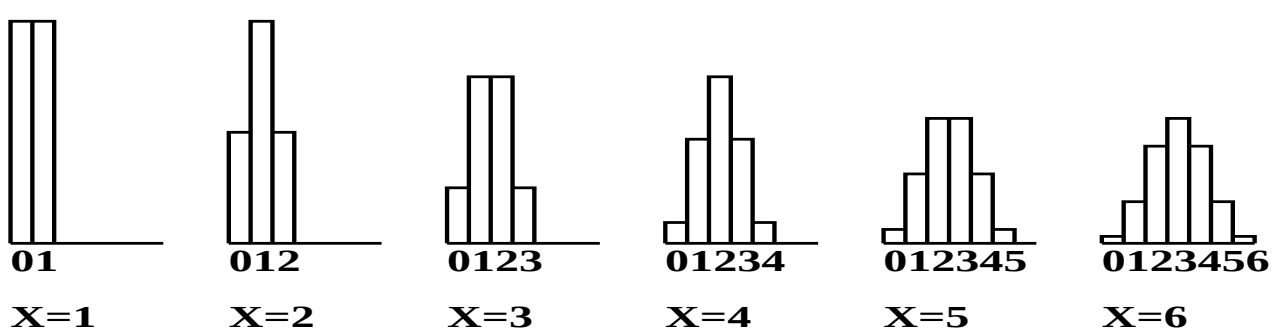
Discrete conditional distributions

Number of successes in a random number of trials

- X is the number showing on a fair die. X fair coins are then tossed, to give a total of Y heads.

$$P\{Y = y | X = x\} \quad = \quad \binom{x}{y} 2^{-x}, \qquad 0 \le y \le x.$$

- This is the conditional distribution of Y given $X = x$.



- Distribution of Y:

$$
\begin{aligned}
P\{Y=y\} &= \sum_{x=1}^{6} P\{Y=y|X=x\} \times P\{X=x\} \\
&= \frac{1}{6} \sum_{x=1}^{6} \binom{x}{y} 2^{-x}, \qquad 0 \le y \le 6 \\
P\{Y=0\} &= \frac{1}{6}\left[\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^6}\right] = \frac{63}{384}
\end{aligned}
$$

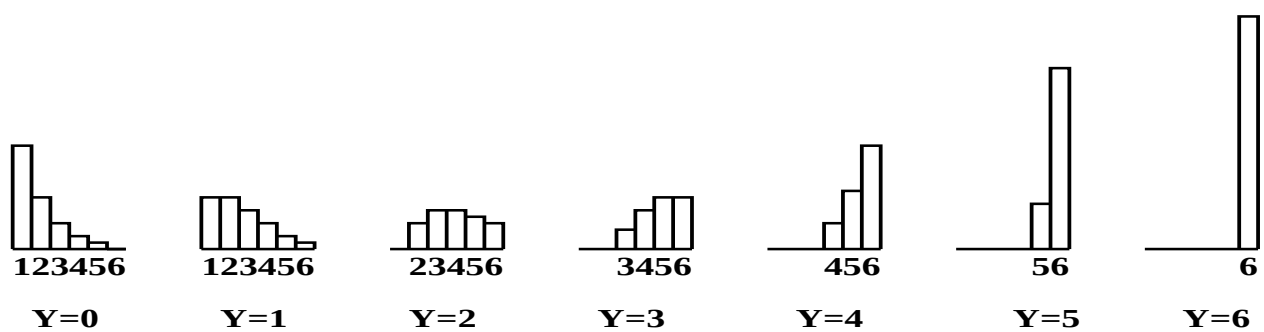
y	0	1	2	3	4	5	6
$P\{Y=y\}$	$\frac{63}{384}$	$\frac{120}{384}$	$\frac{99}{384}$	$\frac{64}{384}$	$\frac{29}{384}$	$\frac{8}{384}$	$\frac{1}{384}$

Rule of average conditional probabilities:

$P\{Y=y\} = \sum_x P\{Y=y|X=x\} \times P\{X=x\}$

- Conditional distribution of X given $Y = y$:

$$
\begin{aligned}
P\{X = x | Y = y\} &= \frac{P\{X = x, Y = y\}}{P\{Y = y\}} \\
&= \frac{P\{Y = y | X = x\}\ P\{X = x\}}{P\{Y = y\}} \\
&= \frac{\binom{x}{y}\ 2^{-x}\ \frac{1}{6}}{\sum_{x=0}^{6} \binom{x}{y}\ 2^{-x}} \qquad 0 \le y \le x \le 6
\end{aligned}
$$



Conditioning Poissons on their sum

- $N_1, \ldots, N_m$ are independent Poissson with parameters $\lambda_1, \ldots, \lambda_m$.
- Find the conditional distribution of $(N_1, \ldots, N_m)$ given that $N_1 + \ldots + N_m = n$.
- When $n_1 + \ldots + n_m = n$,

$$
\begin{aligned}
&P\{N_1 = n_1, \ldots, N_m = n_m | N_1 + \ldots + N_m = n\} \\
&\quad = \frac{\left(e^{-\lambda_1}\lambda_1^{n_1}/n_1!\right)\cdots\left(e^{-\lambda_n}\lambda_m^{n_m}/n_m!\right)}{e^{-(\lambda_1+\ldots+\lambda_m)}(\lambda_1+\ldots+\lambda_m)^n/n!} \\
&\quad = \frac{n!}{n_1!\cdots n_m!}\left(\frac{\lambda_1}{\lambda_1+\ldots+\lambda_m}\right)^{n_1}\cdots\left(\frac{\lambda_m}{\lambda_1+\ldots+\lambda_m}\right)^{n_m}
\end{aligned}
$$

- Multinomial!

Conditional expectation given event A

$$E(Y|A) \equiv \sum_y y P\{Y = y|A\}$$

- Example: Y is the number of heads in 4 tosses of a fair coin.
- $A = \{2 \text{ or less heads}\}$.

$$\begin{aligned} P\{Y = y\} &= \binom{4}{y} 2^{-4} = \binom{4}{y}/16 \qquad (0 \le y \le 4) \\ \Rightarrow \quad P\{Y \le 2\} &= (1+4+6)/16 = 11/16 \\ \Rightarrow \quad P\{Y = y|Y \le 2\} &= \binom{4}{y}\Big/11 \qquad (0 \le y \le 2) \end{aligned}$$

and so

$$E(Y|Y \le 2) = \sum_{y=0}^{2} y \binom{4}{y}\Big/11 = (1 \cdot 4 + 2 \cdot 6)/11 = 16/11$$

Properties of conditional expectation

- ALL the basic rules of expectation hold.
  e.g. $E\{X+Y|A\} \;=\; E\{X|A\}+E\{Y|A\}$
- If $A_1, A_2, \ldots$ is a partition of the entire probability space

$$\boxed{E(Y) \;=\; \sum_k E(Y|A_k)\; P\{A_k\}}$$

$$\begin{aligned}
E(Y) &= \sum_y y\; P\{Y=y\} \\
&= \sum_y y \left(\sum_k P\{Y=y|A_k\} P\{A_k\} \right) \\
&= \sum_k \left(\sum_y y P\{Y=y|A_k\} \right) P\{A_k\} \;=\; \sum_k E(Y|A_k)\; P\{A_k\}
\end{aligned}$$

- X is the number showing on a fair die. X fair coins are then tossed, to give a total of Y heads.
- Find $E(Y)$.
- Recall $E(Y) \;=\; \sum_k E(Y|A_k)\; P\{A_k\}$.
- Set $A_k = \{X = k\},\; k = 1, \ldots, 6$.
- Then $E(Y|A_k) = k/2$.

$$\begin{aligned} \Rightarrow \qquad E(Y) &= \left(\frac{1}{2} + \frac{2}{2} \ldots + \frac{6}{2}\right) \Big/ 6 \\ &= (1 + 2 + \ldots + 6)/12 \\ &= 21/12 \\ &= 1.75 \end{aligned}$$

Conditional expectation given the event $X = x$:

$$E(Y|X = x) \equiv \sum_y y P\{Y = y|X = x\}$$

- Nothing changes, just the conditioning event.
- $\Rightarrow$ we again have

$$E(Y) = \sum_x E(Y|X = x)\ P\{X = x\}$$

- Note that $E(Y|X = x)$ is now <u>a function of x</u>. Write

$$g(x) \equiv E(Y|X = x)$$

Conditional expectation of Y given X:

$$E(Y|X) \equiv g(X)$$

$$(\ E(X|X) \equiv X\ )$$

$$E(Y|X) \equiv g(X)$$

where $g(x) = E(Y|X = x)$.

- X is the number showing on a fair die. X fair coins are then tossed, to give a total of Y heads.
- $g(x) = E(Y|X = x) = x/2$.
- $E(Y|X) = X/2$.

$$E(Y) = E[E(Y|X)]$$

- $E(Y) = E[E(Y|X)] = E[X/2] = 3.5/2 = 1.75$.

$$\boxed{E(Y) \;=\; E[E(Y|X)]}$$

$$\begin{aligned}
E[E(Y|X)] &= E\left[g(X)\right] \\
&= \sum_x g(x)\; P\{X = x\} \\
&= \sum_x E(Y|X = x)\; P\{X = x\} \\
&= \sum_x \left[\sum_y y\; P\{Y = y|X = x\}\right]\; P\{X = x\} \\
&= \sum_x \sum_y y\; P\{Y = y, X = x\} \\
&= \sum_y y\; \sum_x P\{Y = y, X = x\} \\
&= \sum_y y\; P\{Y = y\} \quad = \quad E(Y)
\end{aligned}$$

Expectation of a random sum, using $E(Y) = E[E(Y|X)]$

- $X_1, X_2, \ldots$ are (not necessarily independent) random variables, with common, finite mean, $E(X_k) = \mu$.
- N takes values in $\{0, 1, 2, \ldots\}$, independent of all the X_k.
- $S_N \equiv X_1 + \ldots + X_N$.

$$\boxed{E(S_N) = E(X) \cdot E(N).}$$

PROOF: Note

$$\begin{aligned} E[S_N|N = n] &= E[X_1 + \ldots + X_n] = n\mu \\ E[S_N] &= E[E(S_N|N)] = E[N\mu] \\ &= \mu \cdot E[N] \end{aligned}$$

- Result need not be true without independence.

Why we need independence: example

- i.e. Independence between the X_k and N.
- $X_1, X_2, \ldots$ are independent random variables, taking integer values, with common, finite mean, $E(X_k) = \mu = 0$.
- $N = \inf\{n : S_n = 17\}$
- $S_N \equiv X_1 + \ldots + X_N \;\; \Rightarrow S_N \equiv 17$

$$17 \;=\; E(S_N) \neq 0 \;=\; E(X) \cdot E(N)$$

$$\boxed{E(S_N) \;=\; E(X) \cdot E(N).}$$

- Number of children in a family has mean $\lambda = 2.8$.
- Probability that child is son is 0.515.
- What is the expected number of sons?

$$X_k \;=\; \begin{cases} 1 & \text{if child } k \text{ is male} \\ 0 & \text{otherwise} \end{cases} \qquad\qquad E(X_k) \;= 0.515$$

$$N \;=\; \text{Number of children} \qquad\qquad E(N) = 2.8$$

$$S_N \;=\; X_1 + \ldots + X_N \;=\; \text{Number of sons.}$$

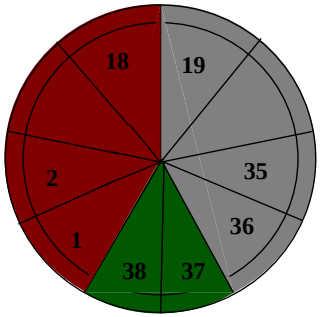
- $E(\text{Number of sons}) \;=\; 0.515 \cdot 2.8 \;=\; 1.442$

Sharkey's casino (Lecture 19)

- N_R reds, N_B blacks, in n spins
- r = proportion of red slots

  b = proportion of black slots

  $1 - r - b$ = proportion of green slots



Find $Corr(N_R, N_B)$

$$
\begin{aligned}
Corr(N_R, N_B) &= \frac{Cov(N_R, N_B)}{\sqrt{Var(N_R)\,Var(N_B)}} \\
&= \frac{-Nrb}{\sqrt{Nr(1-r)}\,\sqrt{Nb(1-b)}} \\
&= \frac{-rb}{\sqrt{r(1-r)}\,\sqrt{b(1-b)}} \qquad (= -0.9 \text{ if } r = b = 18/38)
\end{aligned}
$$

What happens if N is random?

- Condition on N, and use the fixed N computations and $N_R \sim Bin(N, r),\ N_B \sim Bin(N, b)$ $N_R + N_B \sim Bin(N, r+b)$ to compute:

$$
\begin{aligned}
E(N_R N_B|N) &= E(N_R\big|N)E(N_B\big|N) \ + \ Cov(N_R, N_B\big|N) \\
&= N^2 rb \ - \ Nrb \\
\Rightarrow \qquad Cov(N_R, N_B) &= E\big[E(N_R N_B\big|N) - E(N_R|N)E(N_B\big|N)\big] \\
&= rb\big[E(N^2) \ - \ E(N) \ - [E(N)]^2\big] \\
&= rb\big[Var(N) \ - \ E(N)\big]
\end{aligned}
$$

- Similarly,

$$
\begin{aligned}
Var(N_R) &= E\big[E(N_R^2\big|N)\big] \ - \{E[E(N_R|N)]\}^2 \\
&= E\big[Nr(1-r) \ + \ (Nr)^2\big] \ - \ \{E[Nr]\}^2 \\
&= r^2 Var(N) \ + \ r(1-r)E(N) \\
\Rightarrow \quad Var(N_B) &= b^2 Var(N) \ + \ b(1-b)E(N)
\end{aligned}
$$

- Put all this together to get

$$Corr(N_R, N_B)$$

$$= \frac{rb[Var(N) - E(N)]}{\sqrt{r^2 Var(N) + r(1-r)E(N)]\ [b^2 Var(N) + b(1-b)E(N)]}}$$

- If $E(N) = 1,000,\ SD(N) = 100,\ r = b = 18/38$

$$Corr(N_R, N_B) \ = \ 0.81$$

Summary

- $P\{Y = y|A\}, \quad P\{Y = y|X = x\}$
- $E(Y|A) \;=\; \sum_y y \; P\{Y = y|A\}$
- $E(Y) \;=\; \sum_k E(Y|A_k) \; P\{A_k\}$
- $E(Y|X = x) \;=\; \sum_y y \; P\{Y = y|X = x\}$
- $E(Y) \;=\; \sum_x E(Y|X = x) \; P\{X = x\}$
- $E(Y|X) = g(X)$, where $g(x) = E(Y|X = x)$
- $E(Y) \;=\; E[E(Y|X)]$
- $E(S_N) \;=\; E(X) \cdot E(N)$