Lecture 32

$$E(Y) = E[E(Y|X)]$$

Success counts in overlapping trials

Continuous distributions

Conditional densities

Examples

Independence

Discrete conditioning
Summary

- $P\{Y = y|A\}, \quad P\{Y = y|X = x\}$
- $E(Y|A) \;=\; \sum_y y \; P\{Y = y|A\}$
- $E(Y) \;=\; \sum_k E(Y|A_k) \; P\{A_k\}$
- $E(Y|X = x) \;=\; \sum_y y \; P\{Y = y|X = x\}$
- $E(Y) \;=\; \sum_x E(Y|X = x) \; P\{X = x\}$
- $E(Y|X) = g(X)$, where $g(x) = E(Y|X = x)$
- $E(Y) \;=\; E[E(Y|X)]$
- $E(S_N) \;=\; E(X) \cdot E(N)$

Success counts in overlapping trials

- S_n is the number of success in n Bernoulli(p) trials.
- Find $E(S_m|S_n = k)$ for $m \leq n$.

$$
\begin{aligned}
E(S_m \mid S_n = k) &= E(X_1 + \ldots + X_m \mid S_n = k) \\
&= \sum_{j=1}^{m} E(X_j \mid S_n = k) \\
&= \sum_{j=1}^{m} P\{\text{Trial } j \text{ a success} \mid S_n = k) \\
&= \sum_{j=1}^{m} \frac{P\{\text{Trial } j \text{ a success; } k-1 \text{ other successes}\}}{P\{S_n = k\}} \\
&= m \, \frac{p \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} \quad = \quad \frac{mk}{n}
\end{aligned}
$$

Continuation and extension:

- $E(S_m \mid S_n = k) \;=\; mk/n$
- $\Rightarrow \quad E(S_m \mid S_n) = \frac{m}{n} S_n$ – independent of p.
- Alternative approach, using ONLY symmetry.

  - $S_n \;=\; E(S_n|S_n) \;= \sum_{j=1}^{n} E(X_j|S_n) \;=\; nE(X_1|S_n)$.
  - $\Rightarrow E(X_1|S_n) \;=\; E(X_j|S_n) \;=\; S_n/n$.
  - $E(S_m \mid S_n) \;=\; \sum_{j=1}^{m} E(X_j|S_n) \;=\; \frac{m}{n} S_n$

- NOTE: The above uses symmetry (exchangeability) and does not need not independence!

Joint probabilities

- $X_1, X_2, \ldots$, independent, $U[0,1]$, and $N \sim \text{Poiss}(\lambda)$.
- What is the probability that <u>all of</u> $X_1, \ldots, X_N$ are less than t, $t \in [0,1]$?

◦ Set $A_k = \{X_k \leq t\}$ and $I_k = I_{A_k}$.

◦ We want $E(I_1 I_2 \cdots I_N)$.

$$
\begin{aligned}
E(I_1 I_2 \cdots I_N) &= E\left[E\left(I_1 I_2 \cdots I_N \mid N\right)\right] \\
&= E\left[E\left(I_1\right) \cdots E\left(I_N\right)\right] \\
&= E\left[t^N\right] \\
&= \sum_{k=0}^{\infty} t^k e^{-\lambda} \lambda^k / k! \\
&= e^{-\lambda(1-t)}
\end{aligned}
$$

Continuous distributions

- We are familiar with

Joint density	$f_{XY}(x,y)$	$f_{XY}(x,y)dxdy = P\{X \in dx, Y \in dy\}$
Marginal of X	$f_X(x)$	$\int_y f_{XY}(x,y)\ dy$
Marginal of Y	$f_Y(y)$	$\int_x f_{XY}(x,y)\ dx$

- We are looking for a conditional density

$$f_{Y|X}(y|x)\ dy \ = \ P\{Y \in dy \mid X = x\} \ = \ \lim_{dx \to 0} P\{Y \in dy \mid X \in dx\}$$

$$\boxed{f_{Y|X}(y|x) \ = \ \frac{f_{XY}(x,y)}{f_X(x)}}$$

$$\boxed{f_{Y|X}(y|x) \;=\; f_{XY}(x,y)\Big/ f_X(x)}$$

- We defined

$$f_{Y|X}(y|x)\;dy \quad = \quad \lim_{dx\to 0}\; P\{Y \in dy \mid X \in dx\}$$

Consider

$$\begin{aligned} P\{Y \in dy \mid X \in dx\} \quad &= \quad \frac{P\{Y \in dy, \quad X \in dx\}}{P\{X \in dx\}} \\ &\approx \quad \frac{f_{XY}(x,y)\;dxdy}{f_X(x)\;dx} \\ &= \quad \frac{f_{XY}(x,y)}{f_X(x)}\;dy \end{aligned}$$

Independence

- X, Y independent

$$\overset{\text{Definition}}{\Longleftrightarrow} P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\},\ \forall A, B$$

$$\overset{\text{Lecture 30}}{\Longleftrightarrow} \quad f_{XY}(x, y) \ = \ f_X(x) \cdot f_Y(y),\ \forall x, y$$

- Under independence

$$f_{Y|X}(y|x) \ = \ \frac{f_{XY}(x, y)}{f_X(x)} \ = \ \frac{f_X(x) \cdot f_Y(y)}{f_X(x)} \ = \ f_Y(y)$$

$$f_{X|Y}(x|y) \ = \ \frac{f_{XY}(x, y)}{f_Y(y)} \ = \ \frac{f_X(x) \cdot f_Y(y)}{f_Y(y)} \ = \ f_X(x)$$

$$\boxed{f_{XY}(x,y) \;=\; f_{Y|X}(y|x)f_X(x)}$$

- $X \sim \Gamma(2,\lambda)$ and $(Y|X=x) \sim U[0,x]$.
- Joint density of (X,Y).

$$\begin{aligned} f_X(x) &= \lambda^2 x e^{-\lambda x} \qquad x > 0 \\ f_{Y|X}(y|x) &= 1/x \qquad 0 < y < x \\ \Rightarrow \qquad f_{XY}(x,y) &= \lambda^2 e^{-\lambda x} \qquad 0 < y < x \end{aligned}$$

- Marginal density of Y.

$$f_Y(y) \;=\; \int_y^\infty \lambda^2 e^{-\lambda x}\, dx \;=\; \lambda e^{-\lambda y} \qquad y > 0$$

Joint, marginal, and conditional densities

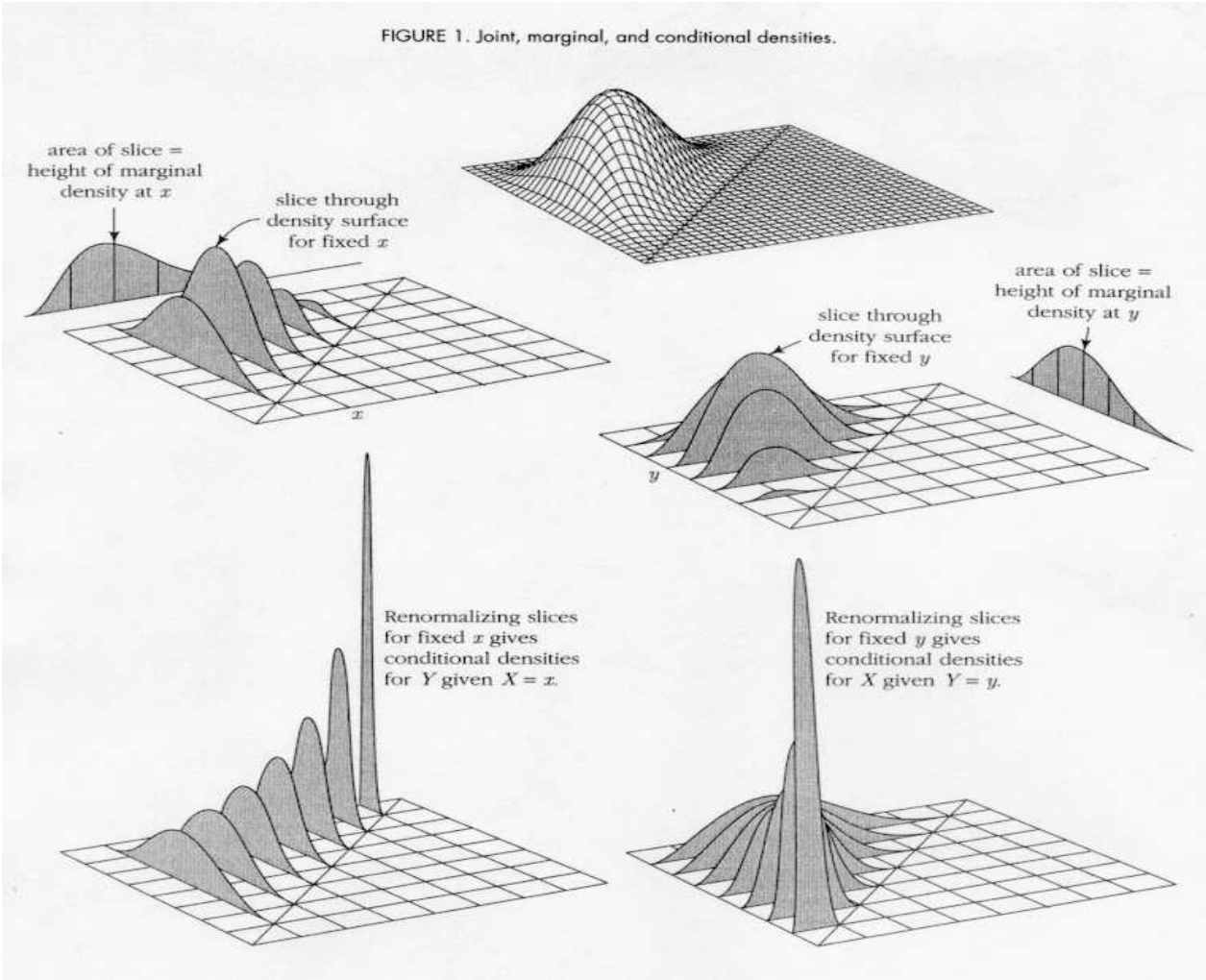
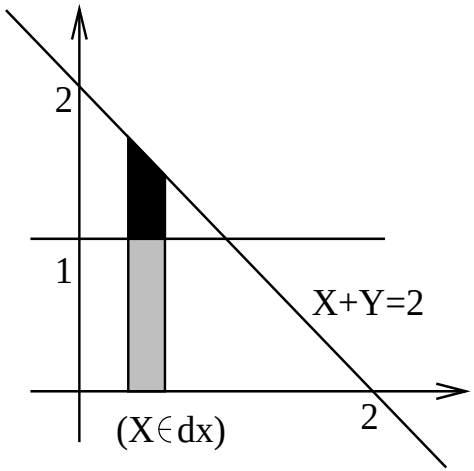


FIGURE 1. Joint, marginal, and conditional densities.

Uniform on a triangle



- Choose a point at random from
  $\{D = (x, y) : 0 \le x, y; 0 \le x + y \le 2\}$
- Joint density:
  $f_{XY}(x, y) = \frac{1}{2}$, if $(x, y) \in D$.
- Marginal of X:
  $f_X(x) = \int_{y=0}^{2-x} \frac{1}{2}\, dy = (2 - x)/2, \qquad 0 \le x \le 2$
- Marginal of Y: $f_Y(y) = (2 - y)/2, 0 \le y \le 2$
- Conditional of of $(Y|X = x)$, for $0 \le y \le 2 - x$:

$$f_{Y|X}(y|x) = f_{XY}(x, y) \Big/ f_X(x) = \tfrac{1}{2}/[(2 - x)/2] = 1/(2 - x)$$

- i.e. $Y \big| X \sim U[0, 2 - X]$

Uniform order statistics

- $U_1, U_2, \ldots, U_n$ independent, $U[0,1]$
- $X = U_{(1)}$ and $Y = U_{(2)}$ are the first two order statistics.

$$f_{XY}(x,y) = \begin{cases} n(n-1)(1-y)^{n-2} & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- Marginal of Y: $f_Y(y) \;=\; n(n-1)y(1-y)^{n-2}, \qquad 0 \le y \le 1$
- Conditional of $X|Y = y$

$$\begin{aligned} f_{X|Y}(x|y) &= f_{XY}(x,y) \Big/ f_Y(y) \\ &= \frac{1}{y} \qquad\qquad 0 \le x \le y \end{aligned}$$

- i.e. $X \Big| Y \sim U[0, Y]$