Lecture 36

Multivariate CLT

Multivariate normal

Covariance and correlation

Bivariate normal

A special construction

Multivariate CLT

- Take independent copies $\hat{X}_1, \hat{X}_2, \ldots$ of a random <u>vector</u> $\hat{X} = (X(1), \ldots, X(d))$
- Vector of means: $\hat{\mu} = (E[X(1)], \ldots, E[X(d)])$
- Matrix of variances and covariances, $\Sigma = (\sigma_{ij})_{i,j=1,\ldots,d}$, where $\sigma_{ij} = Cov(X(i), X(j))$
- We have d different CLT's here, each one stating

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} [X_i(k) - \mu(k)] \;\rightarrow\; N(0, \sigma_{kk}), \qquad k = 1, \ldots, d$$

- What can we say about

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\hat{X}_i - \hat{\mu}]$$

Multivariate Normal

- Write $\widehat{Y} \sim N(\hat{\mu}, \Sigma)$ if

$$f_{\widehat{Y}}(\widehat{y}) \quad = \quad \frac{1}{|\Sigma|^{1/2}(2\pi)^{d/2}} \ \exp\left(-(\hat{y} - \hat{\mu})\Sigma^{-1}(\hat{y}' - \hat{\mu}')/2\right)$$

- From the last lecture (with $\hat{\mu} = 0$) we know that if $X_1, \ldots, X_n \sim N(0, 1)$, independent, and

$$\widehat{Y} \ = \ \widehat{X}A \ + \ \mu \quad \Rightarrow \quad Y \sim N(\mu, \Sigma)$$

  where $\Sigma = A'A$

- Σ is called the covariance matrix of the random variable $\widehat{Y}$, since $\sigma_{ij} = Cov(Y_i, Y_j)$

Multivariate Central Limit Theorem

- If $\hat{X}_1, \hat{X}_2, \ldots$ are independent copies of a random <u>vector</u> $\hat{X} = (X(1), \ldots, X(d))$ with means $\hat{\mu}$ and variances-covariance matrix Σ, then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\hat{X}_i - \hat{\mu}] \approx N(0, \Sigma)$$

  for large n.

- Note: The precise form of dependence is lost in the limit. Only covariances remain!

Covariance and correlation

- $Cov(X,Y) \;=\; E\{(X-\mu_X)\,(Y-\mu_Y)\}$
- $Cov(X,Y) \;=\; E\{XY\} - \mu_X\mu_Y$
- $Var(X+Y) \;=\; Var(X) \;+\; Var(Y) \;+\; 2Cov(X,Y)$
- X, Y independent $\Rightarrow\; Cov(X,Y) = 0$
- $Corr(X,Y) \;=\; Cov(X,Y)/\sigma_X\sigma_Y$
- $-1 \leq Corr(X,Y) \leq +1$
- $Cov\left(\sum_j a_j X_j,\; \sum_k b_k Y_k\right) \;=\; \sum_j \sum_k a_j b_k \; Cov(X_j, Y_k)$
- X, Y, Z independent $\Rightarrow\; Cov((X+Z),(Y+Z)) = \sigma_Z^2$

Bivariate normal

- $X, Y \sim N(0,1)$, independent.
- $U = aX + bY,\ V = cX + dY.$
  $\Rightarrow a_{11} = a,\ a_{12} = b,\ a_{21} = c,\ a_{22} = d$
- $Var(U) = a^2 + b^2,\ Var(V) = c^2 + d^2,\ Cov(U,V) = ac + bd$

$$
\begin{aligned}
f_{UV}(u,v) &= \frac{1}{2\pi|A|}\ \exp\left(-(u,v)\left(A'A\right)^{-1}(u,v)/2\right) \\
&= \frac{1}{2\pi|\Sigma|^{1/2}}\ \exp\left(-(u,v)\Sigma^{-1}(u,v)'/2\right)
\end{aligned}
$$

$$
A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \qquad \Sigma = A'A = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}
$$

Rewriting the bivariate normal

- Write

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$$

where $\rho = Corr(X, Y)$. Then

$$\Sigma^{-1} = \frac{1}{\sigma_X^2\sigma_Y^2(1-\rho^2)} \begin{pmatrix} \sigma_Y^2 & -\rho\sigma_X\sigma_Y \\ -\rho\sigma_X\sigma_Y & \sigma_X^2 \end{pmatrix}$$

and so

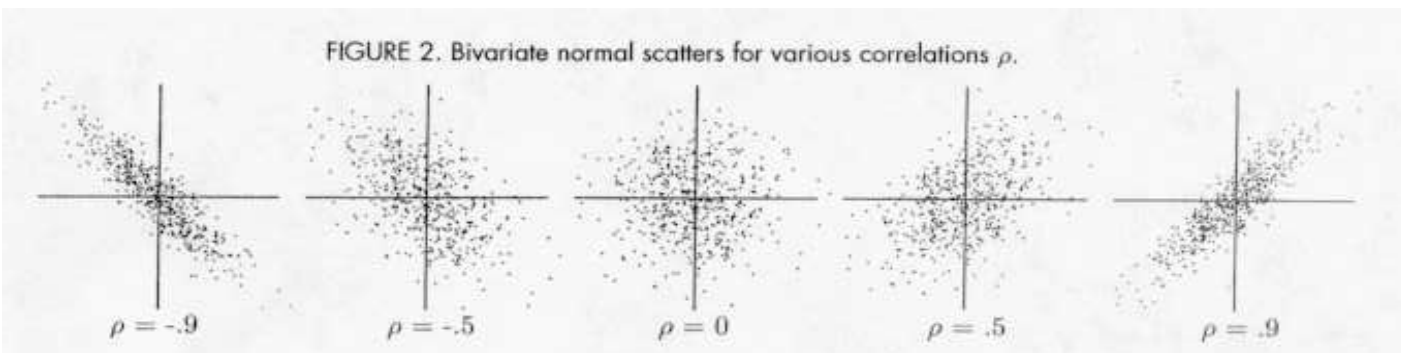
$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$
$$\times \exp\left(\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right)$$

Understanding the bivariate normal

- Take X, Y, with $\mu_X = \mu_Y = 0$, $\sigma_X^2 = \sigma_Y^2 = 1$, $\rho \in (-1, 1)$.

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left[x^2 - 2\rho xy + y^2\right]\right)$$

- A sample scatterplot



- We want to find marginal and conditional distributions, all of which can be done by integration.

Special construction

- Take $X, Z \sim N(0,1)$, independent
- Set $Y = \rho X \; + \; \sqrt{1-\rho^2}\, Z$
- Note: $E(X) = E(Y) = 0$,
  $Var(X) = Var(Y) = 1, \; Corr(X,Y) = \rho$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \cdot \begin{bmatrix} X \\ Z \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

- $\Rightarrow$ (X,Y) has the "standard bivariate normal" distribution
- $\Rightarrow$ all properties about this distribution will follow by studying the pair (X,Y) defined above

Properties of standard bivariate normal

- $X, Z \sim N(0,1)$, independent, and $Y = \rho X \; + \; \sqrt{1-\rho^2}\, Z$
- Marginals: X and Y are marginally $N(0,1)$
- Conditional of $Y|X$:
  - Given $X = x$, we have $Y \sim N(\rho\, x, 1-\rho^2)$
  - $\Rightarrow \;\; E(Y|X) = \rho\, X, \;\; Var(Y|X) = 1-\rho^2.$
- Conditional of $X|Y$: By symmetry of the joint density
  - Given $Y = y$, we have $X \sim N(\rho\, y, 1-\rho^2)$
  - $\Rightarrow \;\; E(X|Y) = \rho\, Y, \;\; Var(X|Y) = 1-\rho^2.$
- Independence $\iff Corr(X,Y) = \rho = 0$

General bivariate normal

- X and Y have a general normal distribution with parameters μ_X, μ_Y, σ_X^2, σ_Y^2 and ρ $\iff$ the normalised random variables

$$U = \frac{X - \mu_X}{\sigma_X} \qquad V = \frac{Y - \mu_Y}{\sigma_Y}$$

  have a standard bivariate normal distribution with correlation ρ where

$$\rho = Corr(U, V) = Corr(X, Y)$$

- Note:

$$X = \sigma_X U + \mu_X, \qquad Y = \sigma_Y V + \mu_Y$$

- Proofs by standard transformation calculations.

Conditional distributions

- X and Y have a general normal distribution with parameters μ_X, μ_Y, σ_X^2, σ_Y^2 and ρ, what is the <u>conditional</u> distribution of $X|Y$?

- Can be done usual way, by dividing joint density of X and Y by marginal of Y. But this is a mess.

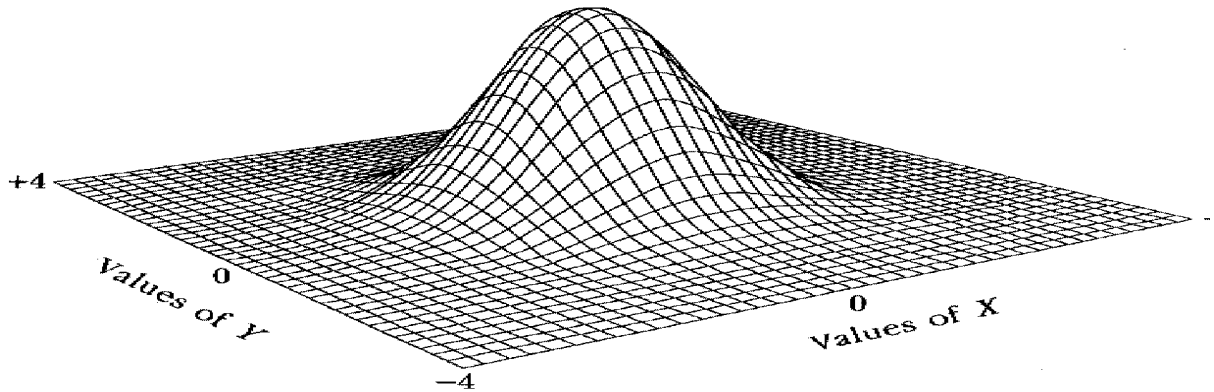
- Better to use special construction for standard bivariate and then linear transformation.

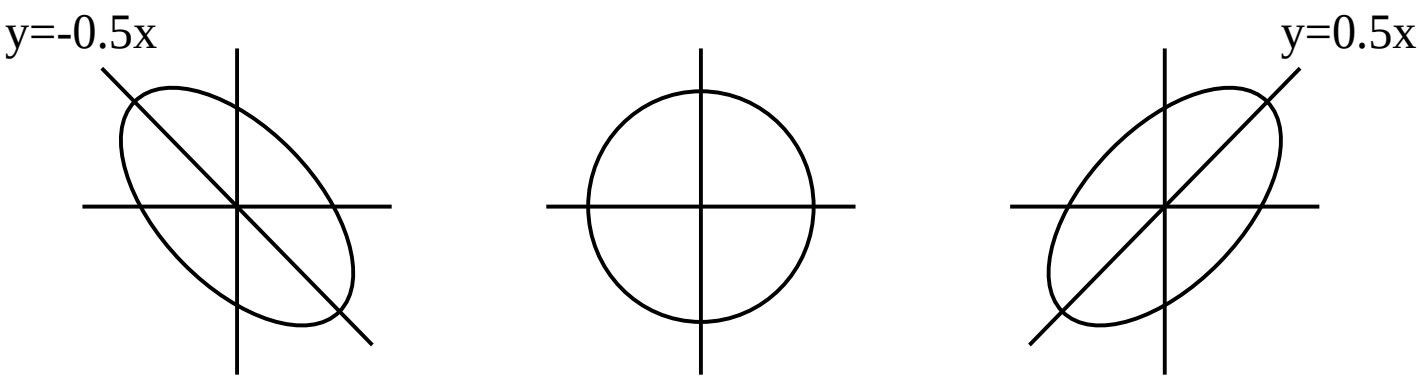
- Result is:

$$X|Y \sim N\left(\mu_X + \frac{\rho\sigma_X}{\sigma_Y}(Y - \mu_Y), \quad \sigma_X^2\left(1 - \rho^2\right)\right)$$

Normal graphics

FIGURE 1. Perspective plot of the joint density of X and Y.





ρ=-0.5 ρ=0 ρ=0.5