Lecture 35

Revision of one dimensional transformations

n dimensional transformations

Linear transformations

Sums of random variables (again)

Multivariate normal

1-1, differentiable transformations

- Take $g : \Re \to \Re$, 1-1 and differentiable.
- Set $Y = g(X)$, and let f_X be the density of X.

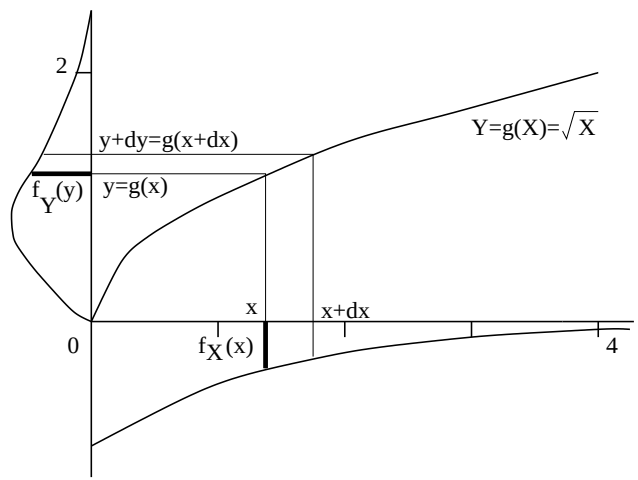
$$\boxed{\begin{aligned} f_Y(y) &= f_X(x) \Big/ \left| \frac{dy}{dx} \right| \\ &= f_X(g^{-1}(y)) \Big/ \left| g'\left(g^{-1}(y)\right) \right| \end{aligned}}$$

where

- g must be 1-1 and differentiable
- The equation $y = g(x)$ must be solved for x in terms of y (i.e. $x = g^{-1}(y)$), and this value of x substituted into $f_X(x)$ and dy/dx.

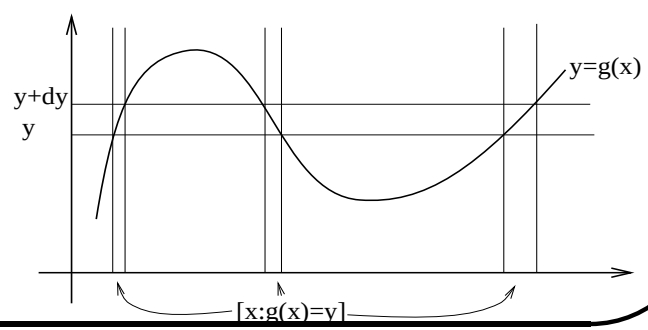
Transforming densities and infinitesimals

$$f_Y(y) \;=\; f_X(x) \Big/ \left| \frac{dy}{dx} \right|$$



Many to one functions

$$f_Y(y) \;=\; \sum_{x:g(x)=y} f_X(x) \Big/ \left| \frac{dy}{dx} \right|$$



1-1, differentiable, n-dimensional, transformations

- Transform $\hat{X} = (X_1, \ldots, X_n)$ to $\hat{Y} = (Y_1, \ldots, Y_n)$

$$\hat{Y} = \hat{g}(\hat{X}), \qquad \hat{X} = \hat{h}(\hat{Y})$$

- In detail:

$$\begin{aligned} Y_1 &= g_1(X_1, \ldots, X_n) & X_1 &= h_1(Y_1, \ldots, Y_n) \\ Y_2 &= g_2(X_1, \ldots, X_n) & X_2 &= h_2(Y_1, \ldots, Y_n) \\ &\ldots\ldots & &\ldots\ldots \\ Y_n &= g_n(X_1, \ldots, X_n) & X_n &= h_n(Y_1, \ldots, Y_n) \end{aligned}$$

- Note: Remain with identical number of random variables after the transformation.

- $\hat{X}$ has density $f_{\hat{X}}(\hat{x}) = f_{\hat{X}}(x_1, \ldots, x_n)$.

$$
\begin{aligned}
P\{\hat{Y} \in A\} &= \int \ldots \int_{\hat{g}(\hat{X}) \in A} f_{\hat{X}}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \\
&= \int \ldots \int_{\hat{y} \in A} f_{\hat{X}}\left(\hat{h}(\hat{y})\right) \, |J(\hat{y})| \, dy_1 \ldots dy_n \\
&\qquad \left(\hat{y} = \hat{g}(\hat{x}),\ \hat{x} = \hat{h}(\hat{y})\right)
\end{aligned}
$$

$$
\boxed{f_{\hat{Y}}(\hat{y}) \;=\; |J(\hat{y})| \; f_{\hat{X}}\left(\hat{h}(\hat{y})\right)}
$$

where

$$
J(y_1, \ldots, y_n) \;=\; \det \begin{pmatrix} \frac{\partial h_1}{\partial y_1} & \ldots & \frac{\partial h_1}{\partial y_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial h_n}{\partial y_1} & \ldots & \frac{\partial h_n}{\partial y_n} \end{pmatrix}
$$

Example 1:

- $X, Y \sim \Gamma(r, \lambda)$, independent.
- $U = X + Y,\ V = X/Y.$

- $f_{XY}(x, y) = f_X(x) f_Y(y) = \lambda^{2r}(xy)^{r-1} e^{-\lambda(x+y)}/[\Gamma(r)]^2$
- $u = g_1(x, y) = x + y, \quad v = g_2(x, y) = x/y.$
- $x = h_1(u, v) = uv/(1 + v), \quad y = h_2(u, v) = u/(1 + v).$

$$
\begin{aligned}
J(u, v) &= \det \begin{pmatrix} \frac{\partial}{\partial u}\left(\frac{uv}{1+v}\right) & \frac{\partial}{\partial v}\left(\frac{uv}{1+v}\right) \\ \\ \frac{\partial}{\partial u}\left(\frac{u}{1+v}\right) & \frac{\partial}{\partial v}\left(\frac{u}{1+v}\right) \end{pmatrix} = \det \begin{pmatrix} \frac{v}{1+v} & \frac{u}{(1+v)^2} \\ \\ \frac{1}{1+v} & \frac{-u}{(1+v)^2} \end{pmatrix} \\
&= \frac{-u}{(1 + v)^2}
\end{aligned}
$$

Example 1: Con't

- $f_{XY}(x,y) = f_X(x)f_Y(y) = \lambda^{2r}(xy)^{r-1}e^{-\lambda(x+y)}/[\Gamma(r)]^2$
- $x = h_1(u,v) = uv/(1+v), \quad y = h_2(u,v) = u/(1+v).$

$$
\begin{aligned}
f_{U,V}(u,v) &= |J(u,v)|\ f_{XY}\left(h_1(u,v), h_2(u,v)\right) \\
&= \begin{cases} \frac{1}{[\Gamma(r)]^2}\ \frac{u}{(1+v)^2}\lambda^{2r}\left(\frac{u^2v}{(1+v)^2}\right)^{r-1} e^{-\lambda u} & 0 < \frac{uv}{1+v} < \infty \\ & 0 < \frac{u}{1+v} < \infty \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{\lambda^{2r}u^{2r-1}e^{-\lambda u}}{\Gamma(2r)}\ \frac{v^{r-1}}{B(r,r)(1+v)^{2r}} & 0 < u, v < \infty \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

- i.e. $U \sim \text{Gamma}(2r)$, and U and V are independent.

Linear transformations

- $\hat{Y}' = A\hat{X}'$, where

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

- Write $|A|$ for $|\det(A)|$, and $B = A^{-1}$ for the inverse matrix.

$$f_{\hat{Y}}(\hat{y}) = \frac{1}{|A|} f_{\hat{X}}\left(A^{-1}\hat{y}'\right)$$

- If $n = 2$, $\hat{X} = (X, Y)$ and $\hat{Y} = (U, V)$, we have

$$\begin{aligned} f_{UV}(u, v) &= \frac{1}{|A|} f_{XY}\left(\frac{a_{22}u - a_{12}v}{|A|}, \frac{a_{11}v - a_{21}u}{|A|}\right) \\ &= \frac{1}{|A|} f_{XY}\left(b_{11}u + b_{12}v,\ b_{21}u + b_{22}v\right) \end{aligned}$$

$(b_{11} = a_{22}/|A|,\ b_{12} = -a_{12}/|A|,\ b_{21} = -a_{21}/|A|,\ b_{22} = a_{11}/|A|)$

Example 2:

- $X, Y \sim N(0,1)$, independent.
- $U = X + Y,\ V = X - Y.$

○ $f_{XY}(x,y) = f_X(x) f_Y(y) = e^{-(x^2+y^2)/2}/(2\pi)$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad |A| = 2 \qquad B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{|A|}\, f_{XY}\left(b_{11}u + b_{12}v,\ b_{21}u + b_{22}v\right) \\ &= \frac{1}{|2|}\,\frac{1}{2\pi}\, e^{-\frac{1}{2}[(u+v)^2/4\ +\ (u-v)^2/4]} \\ &= \frac{1}{\sqrt{2\pi(2)}} e^{-\frac{1}{2}u^2/2}\ \frac{1}{\sqrt{2\pi(2)}} e^{-\frac{1}{2}v^2/2} \end{aligned}$$

- $U \sim N(0,2),\ V \sim N(0,2)$, <u>independently</u>.

Sums of random variables (again)

- (X, Y) have joint density $f_{XY}(x, y)$.
- $U = X + Y,\ V = X.$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad |A| = 1 \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{aligned} f_{UV}(u, v) &= \frac{1}{|A|}\, f_{XY}\left(b_{11}u + b_{12}v,\ b_{21}u + b_{22}v\right) \\ &= f_{XY}\left(v, u - v\right) \end{aligned}$$

- Marginal density of $U = X + Y$ (Convolution formula)

$$f_{X+Y}(u) = \int_v f_{XY}\left(v, u - v\right)\ dv$$

Last example:

- $X_1, \ldots, X_d \sim N(0,1)$, independent.

$$f_{\hat{X}}(\hat{x}) = \prod_{k=1}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_k^2} = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2}(x_1^2 + \ldots + x_d^2)\right)$$

$$= \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2}\hat{x}\hat{x}'\right)$$

- $\hat{Y} = \hat{X}A.$

$$f_{\hat{Y}}(\hat{y}) = \frac{1}{|A|(2\pi)^{d/2}} \exp\left(-\left(\hat{y}A^{-1}\right)\left(\hat{y}A^{-1}\right)'/2\right)$$

$$= \frac{1}{|A|(2\pi)^{d/2}} \exp\left(-\hat{y}A^{-1}(A^{-1})'\hat{y}'/2\right)$$

$$= \frac{1}{|A|(2\pi)^{d/2}} \exp\left(-\hat{y}\left(A'A\right)^{-1}\hat{y}'/2\right)$$

Rewriting....

$$
\begin{aligned}
f_{\hat{Y}}(\hat{y}) &= \frac{1}{|A|(2\pi)^{d/2}} \exp\left(-\hat{y}\,(A'A)^{-1}\,\hat{y}'/2\right) \\
&= \frac{1}{|\Sigma|^{1/2}(2\pi)^{d/2}} \exp\left(-\hat{y}\Sigma^{-1}\hat{y}'/2\right)
\end{aligned}
$$

where $\Sigma \equiv A'A$.

- Σ is called the covariance matrix of the random variable $\hat{Y}$, since,

$$
\Sigma = \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1d} \\ \dots & \dots & \dots \\ \sigma_{d1} & \dots & \sigma_{dd} \end{pmatrix}
$$

$$
\text{where} \qquad \sigma_{ij} = Cov(Y_i, Y_j)
$$

- Details: Next lecture