Lecture 38

Multivariate normal

Non-normal normals

Rotations

χ^2 distribution (again)

Multivariate Normal

- Write $Y \sim N(\mu, \Sigma)$ if

$$f_{\widehat{Y}}(\widehat{y}) \quad = \quad \frac{1}{|\Sigma|^{1/2}(2\pi)^{d/2}} \ \exp\left(-(\hat{y} - \hat{\mu})\Sigma^{-1}(\hat{y}' - \hat{\mu}')/2\right)$$

- From the last lecture (with $\hat{\mu} = 0$) we know that if $X_1, \ldots, X_n \sim N(0, 1)$, independent, and

$$\hat{Y} \ = \ A\hat{X} \ + \ \mu \quad \Rightarrow \quad Y \sim N(\mu, \Sigma)$$

  where $\Sigma = AA'$

- Σ is called the covariance matrix of the random variable $\widehat{Y}$, since $\sigma_{ij} = Cov(Y_i, Y_j)$

Multivariate normal – Properties

1: Marginals are normal

- Partition $\hat{X}$ into
  $\hat{X} = ((X_1, \ldots, X_m), (X_{m+1}, \ldots, X_d) \equiv (\hat{X}_1, \hat{X}_2)$
- Partition $\hat{\mu}$ into $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2)$
- Partition Σ into

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

- Note: Σ_{11} is $m \times m$, $\Sigma_{12} = \Sigma_{21}'$ is $m \times (d-m)$, etc.

$$\hat{X}_1 \sim N\left(\hat{\mu}_1, \Sigma_{11}\right), \quad \hat{X}_2 \sim N(\hat{\mu}_2, \Sigma_{22})$$

2: Conditionals are normal

- Partition as before. In particular

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

$$\boxed{\hat{X}_1 \Big| \hat{X}_2 \sim N\left(\hat{\mu}_1 + \Sigma_{11}^{-1}\Sigma_{12}(\hat{X}_2 - \hat{\mu}_2)',\ \Sigma_{11} - \Sigma_{21}\Sigma_{22}^{-1}\Sigma_{12}\right)}$$

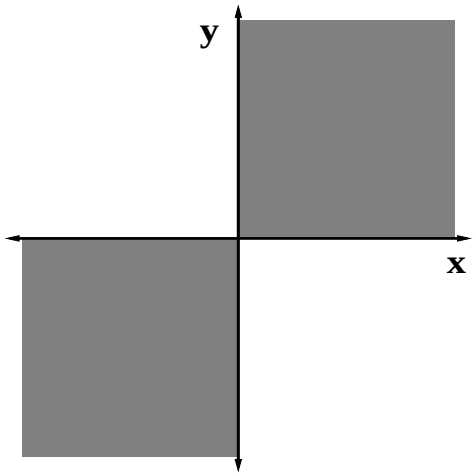
3: Linear combinations remain normal

$$\boxed{\hat{X} \sim N(\hat{\mu}, \Sigma) \ \Rightarrow\ A\hat{X}' + \hat{m} \sim N(\hat{m} + A\hat{\mu}', A\Sigma A')}$$

Non-normal normals

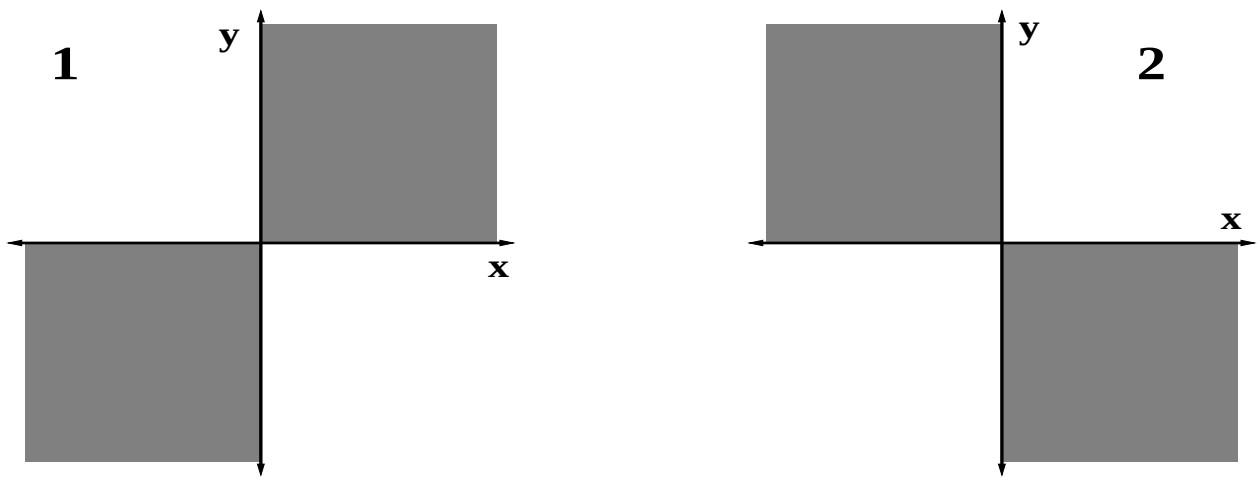
- (X, Y) have the density

$$f_{XY}(x,y) = \begin{cases} \frac{2}{2\pi} e^{-(x^2+y^2)/2} & \text{if } xy \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



- (X, Y) are <u>not</u> bivariate normal
- They <u>are</u> marginally normal. e.g. If $x \geq 0$

$$\begin{aligned} f_X(x) &= \int_0^\infty \frac{2}{2\pi} e^{-(x^2+y^2)/2}\, dy \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, 2 \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \end{aligned}$$



- Can do the same without calculus by "unfolding" mass
- In case 1, X and Y are positively correlated.
- In case 2, X and Y are negatively correlated.
- Can do the same with correlated variables, by starting with general Gaussian density where we had standard.

Rotations: From $\rho = 0$ to $\rho \neq 0$ and back again

- Take $U, V \sim N(0,1)$, independent.
- i.e. (U, V) is bivariate normal with $\rho = 0$.
- Choose and angle $\theta \in [0, 2\pi]$. Set

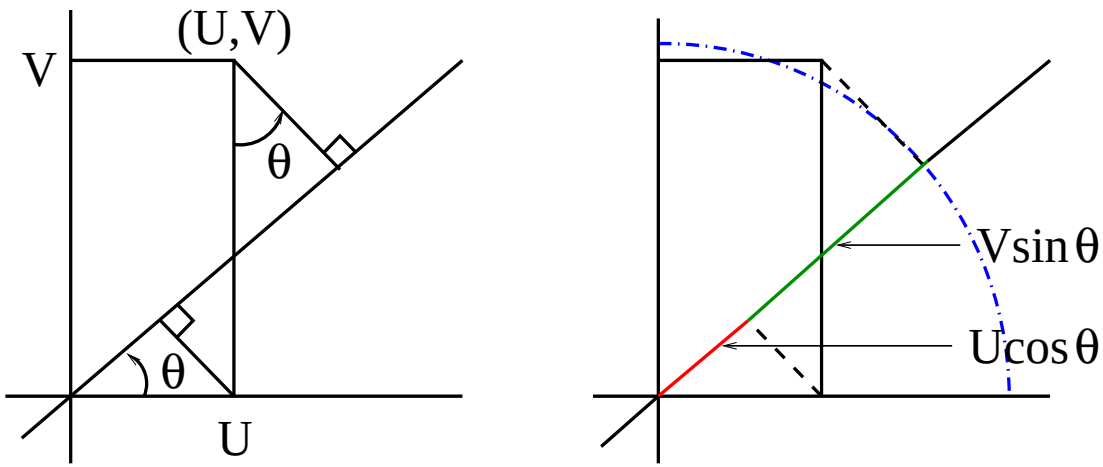
$$\begin{bmatrix} X \\ V \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} U \\ V \end{bmatrix} = A \cdot \begin{bmatrix} U \\ V \end{bmatrix}$$

Since

$$\Sigma \equiv AA' = \begin{bmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{bmatrix}$$

(X, V) is bivariate normal, with zero means and unit variances, and $Corr(X, V) = \sin\theta$

The geometry of correlation



- Rotational symmetry preserves the $N(0,1)$ marginals

$$\begin{bmatrix} X \\ V \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} U \\ V \end{bmatrix} = A \cdot \begin{bmatrix} U \\ V \end{bmatrix}$$

"Back again"

- Suppose $\widehat{X} = (X_1, \ldots, X_n)$ are multivariate normal with zero means, and arbitrary covariance matrix Σ. Density

$$f_X(x) \quad = \quad \frac{1}{|\Sigma|^{1/2}(2\pi)^{n/2}} \ \exp\left(-x'\Sigma^{-1}x/2\right)$$

- Find A such that $A'\Sigma^{-1}A = I$, the identity matrix
- Set $\widehat{Y} \ = \ A \cdot \widehat{X}$
- The Jacobean of the transformation is $|A^{-1}| = |\Sigma|^{1/2}$

$$f_Y(y) = |\Sigma|^{1/2} \ \frac{1}{|\Sigma|^{1/2} \ (2\pi)^{n/2}} \ e^{-(Ay)'\Sigma^{-1}(Ay)/2} = \frac{1}{(2\pi)^{n/2}} \ e^{-y'y/2}$$

- i.e. $\widehat{Y}$ is made up of n independent, standard normals.
- Application: "Factor analysis"

The χ^2 distribution

- Take $X \sim N(0,1)$
- Set $Y = h(X) = X^2$, $\Rightarrow$ $X = g(Y) = \sqrt{Y}$

$$\begin{aligned} f_Y(y) &= 2\,|g'(g(y))|\; f_X(g(y)) \\ &= 2\,\frac{1}{2}y^{-1/2}\; f_X(\sqrt{y}) \\ &= \frac{1}{\sqrt{2\pi}}\; y^{-1/2}\; e^{-y/2} \qquad y > 0. \end{aligned}$$

- Gamma(r,λ) density: $\lambda\, e^{-\lambda t}\,(\lambda t)^{r-1}/\Gamma(r)$
- i.e. $\chi^2 \;\equiv\;$ Gamma$(\frac{1}{2},\frac{1}{2})$ $\qquad\qquad \Rightarrow\; \Gamma(\frac{1}{2}) = \sqrt{\pi}$
- $\Rightarrow$ Sum of n independent χ^2 variables (χ^2_n) has a Gamma$(\frac{n}{2},\frac{1}{2})$ distribution

Cont'd

- $X_1, \ldots, X_n$ idependent, $N(0,1)$ and $Y = X_1^2 + \ldots + X_n^2 \quad \Rightarrow$

$$f_Y(y) \;=\; \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}\; y^{\frac{1}{2}n-1} e^{-y/2}$$

- Homework exercise: Compute this directly (without worrying about the constant) using the spherical symmetry of the n-dimensional multi-normal distribution.
- Hint: See Pitman, p365