Lecture 28

Transformations for densities

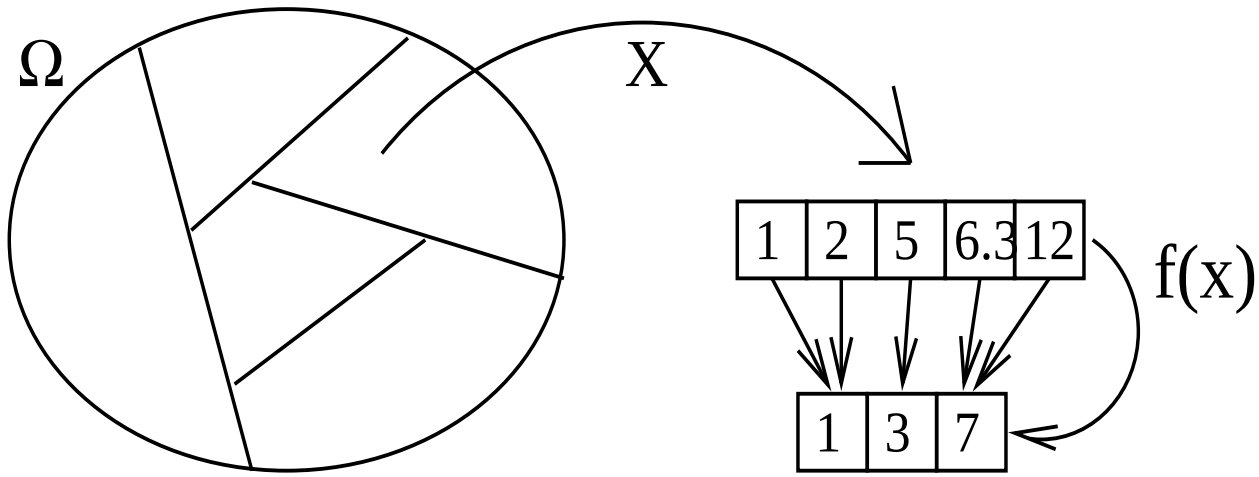
Linear transformations

1-1 differentiable functions

General transformations

Expectation of a function

Transformations for discrete random variables



$$Y = g(X) \quad \Rightarrow \quad p_Y(y) = \sum_{x:g(x)=y} p_X(x)$$

Linear transformation of a $U[0,1]$

- Take $X \sim U[0,1]$, so that

$$f_X(x) \;=\; \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

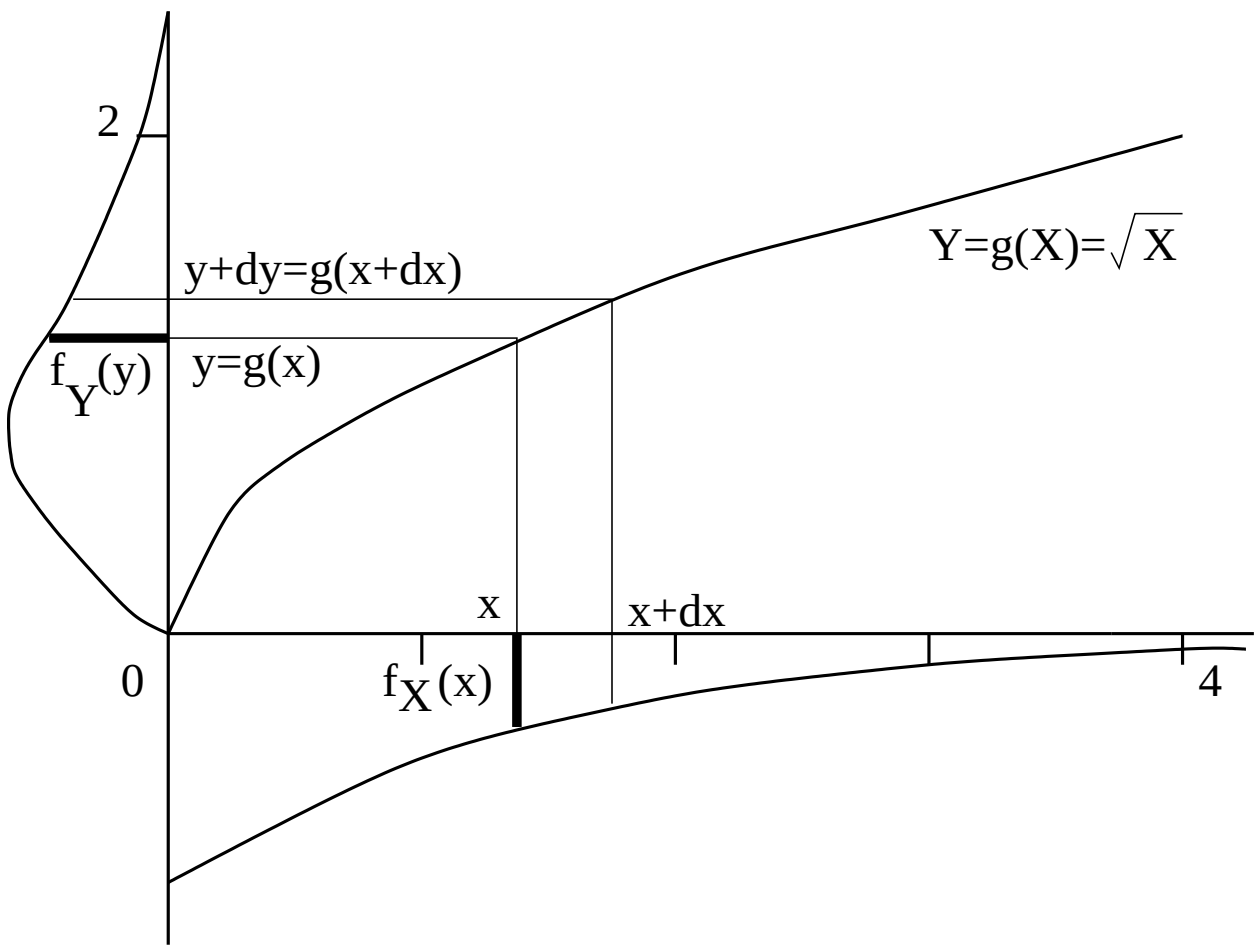
- Take $a > 0$ and set $Y = aX + b$. "Clearly" $Y \sim U[b, b+a]$:

$$f_Y(y) \;=\; \begin{cases} 1/|a| & b < y < b+a \\ 0 & \text{otherwise} \end{cases}$$

- The above is true also if $a < 0$.
- Bottom line:

$$f_Y(y) \;=\; f_{aX+b}(y) \;=\; \frac{1}{|a|}\, f_X\left(\frac{y-b}{a}\right)$$

Transforming densities and infinitesimals



1-1, differentiable transformations: Via infinitesimals

- Take $g : \Re \to \Re$, 1-1 and differentiable.
- Set $Y = g(X)$, and let f_X be the density of X.
- We need to find $P\{Y \in dy\} \equiv f_Y(y)dy$ in terms of f_X.
- From the previous diagram, comparing areas,

$$
\begin{aligned}
P\{Y \in dy\} = P\{X \in dx\} \quad &\Longleftrightarrow \quad f_Y(y)dy = f_X(x)dx \\
&\Longleftrightarrow \quad f_Y(y) = f_X(x)\,\frac{dx}{dy} \\
&\Longleftrightarrow \quad f_Y(y) = f_X(x)\Big/\frac{dy}{dx}
\end{aligned}
$$

where $dy/dx = g'(x)$ and $x = g^{-1}(y)$

$$Y = g(X) \;\Rightarrow\; f_Y(y) = f_X(x) \Big/ \left|\frac{dy}{dx}\right|$$

where

- g must be 1-1 and differentiable
- The equation $y = g(x)$ must be solved for x in terms of y (i.e. $x = g^{-1}(y)$), and this value of x substituted into $f_X(x)$ and dy/dx.

$$f_Y(y) = f_X(g^{-1}(y)) \Big/ \left|g'\left(g^{-1}(y)\right)\right|$$

Square root of an exponential

- $g(x) = \sqrt{x}, \quad f_X(x) = e^{-x},\ x > 0.$
- Note g is 1-1 and differentiable, and range of Y is $[0, \infty]$.
- Since $x = y^2$, inverse function is $g^{-1}(y) = y^2$.
- Compute derivatives:

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

- Substitute into $f_Y(y) = f_X(x)/|dy/dx|$

$$f_Y(y) = e^{-x} \Big/ \frac{1}{2\sqrt{x}}$$

- Eliminate x from the right hand side.

$$f_Y(y) = e^{-y^2} \Big/ \frac{1}{2\sqrt{y^2}} = 2ye^{-y^2}, \qquad 0 < y < \infty.$$

Logarithm of a uniform

- $g(x) = -\lambda^{-1}\log(x), \quad f_X(x) = 1,\ 0 < x < 1.$
- g is 1-1, differentiable, decreasing. Range of Y is $[0, \infty]$.
- Inverse function is $g^{-1}(y) = e^{-\lambda y}$.
- Compute derivatives:

$$\frac{dy}{dx} \;=\; \frac{d}{dx}\left(-\lambda^{-1}\log(x)\right) \;=\; \frac{-1}{\lambda\ x}$$

- Substitute into $f_Y(y) = f_X(x)/|dy/dx|$

$$f_Y(y) \;=\; f_X(x)/|dy/dx| \;=\; 1\Big/\left|\frac{-1}{\lambda\ x}\right| \;=\; \lambda\ x$$

- Eliminate x from the right hand side.

$$f_Y(y) \;=\; \lambda e^{-\lambda\ y}, \qquad 0 < y < \infty.$$

Alternative derivation of transformation formula

- $Y = g(X)$, g is 1-1 and differentiable and increasing.
- Since $f_Y = F_Y'$, it is enough to find F_Y.

$$
\begin{aligned}
F_Y(y) \;\equiv\; P\{Y \le y\} &= P\{g(X) \le y\} \\
&= P\{X \le g^{-1}(y)\} \;=\; F_X(g^{-1}(y)) \\
\Rightarrow \quad f_Y(y) \;=\; \frac{d}{dy}F_Y(y) &= \frac{d}{dy}F_X(g^{-1}(y)) \\
&= F_X'(g^{-1}(y))\;\frac{d}{dy}\left(g^{-1}(y)\right) \\
&= f_X(g^{-1}(y))\;\frac{d}{dy}\left(g^{-1}(y)\right) \\
&= f_X(g^{-1}(y)) \Big/ \left|g'\left(g^{-1}(y)\right)\right|
\end{aligned}
$$

- Ditto for g decreasing.

Application <u>without</u> the formula

Square root of an exponential (again)

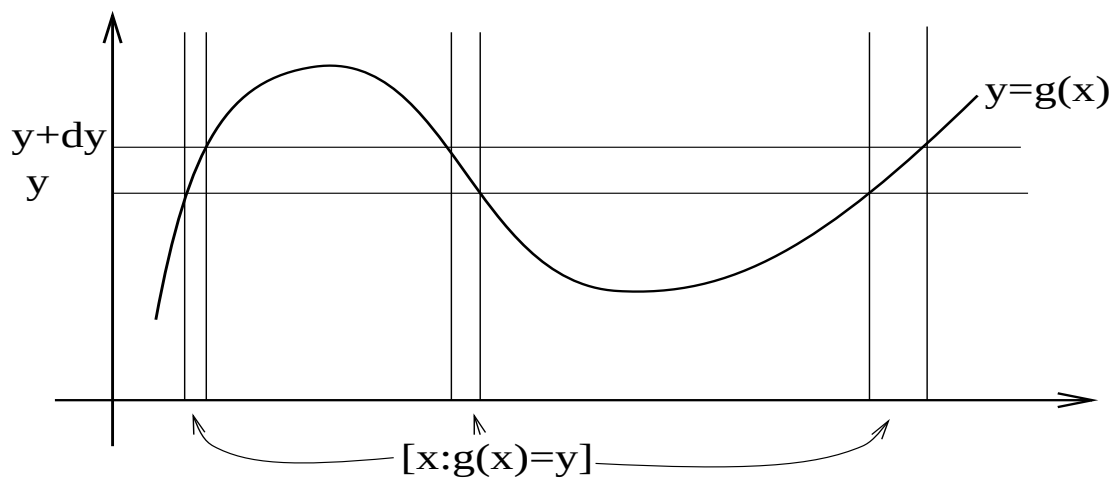
- $g(x) = \sqrt{x}, \quad f_X(x) = e^{-x}, \; x > 0.$
- Note g is 1-1 and differentiable, and range of Y is $[0, \infty]$.
- Recall $F_X(x) = 1 - e^{-x}$.

$$\begin{aligned} F_Y(y) \;\equiv\; P\{Y \le y\} &= P\{\sqrt{X} \le y\} \\ &= P\{X \le y^2\} \\ &= 1 - e^{-y^2} \end{aligned}$$

- Differentiate

$$f_Y(y) \;=\; 2ye^{-y^2}, \qquad 0 < y < \infty.$$

Many to one functions



Since we now have

$$P\{Y \in dy\} \;=\; \sum_{x:g(x)=y} P\{X \in dx\}$$

the rule becomes

$$f_Y(y) \;=\; \sum_{x:g(x)=y} f_X(x) \Big/ \left|\frac{dy}{dx}\right|$$

Density of the square

- $X \sim f_X(x)$, and $Y = X^2$.
- $y = x^2 \;\Rightarrow\; dy/dx = 2x$
- $Y = y \;\Rightarrow\; X = +\sqrt{y}$ or $X = -\sqrt{y}$.

$$\begin{aligned} f_Y(y) &= \sum_{x:x=\pm\sqrt{y}} f_X(x) \Big/ \left|\frac{dy}{dx}\right| \\ &= \left[f_X(+\sqrt{y}) + f_X(-\sqrt{y})\right]/2\sqrt{y} \end{aligned}$$

Example: $X \sim N(0,1)$, $Y = X^2$. (χ^2 distribution)

$$f_Y(y) = \left[\frac{1}{\sqrt{2\pi}} e^{-(+\sqrt{y})^2/2} + \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2}\right]/2\sqrt{y} = \frac{y^{-1/2} e^{-y/2}}{\sqrt{2\pi}},$$

$y > 0$

- $X \sim N(0,1)$, $Y = X^2$, $\qquad f_Y(y) \;=\; \frac{y^{-1/2}e^{-y}}{\sqrt{2\pi}}$
- Gamma distribution for non-integral n
  $f_T(t) = \lambda\, e^{-\lambda t}\,(\lambda t)^{n-1}/\Gamma(n) \qquad \Gamma(n) \;\equiv\; \int_0^\infty e^{-t}t^{n-1}\,dt.$
- i.e. X^2 has a Gamma$(\frac{1}{2},\frac{1}{2})$ distribution
- i.e. χ^2 distribution is a special case of Gamma
- $\Gamma(\frac{1}{2}) \;=\; \sqrt{\pi}$
- $\Rightarrow$ Sum of n independent χ^2 variables (χ^2_n) has a Gamma$(\frac{n}{2},\frac{1}{2})$ distribution
- $X_1,\ldots,X_n$ idependent, $N(0,1)$ and $Y = X_1^2+\ldots+X_n^2 \;\Rightarrow$

$$f_Y(y) \;=\; \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}\; y^{\frac{1}{2}n-1}e^{-y/2}$$

Expectation of a function:

$$E\{g(X)\} = \int g(x)\ f_X(x)\ dx$$

- Set $Y = g(X)$. Assume g is 1-1 and differentiable.

$$\begin{aligned} E\{g(X)\} = E\{Y\} &= \int y\ f_Y(y)\ dy \\ &= \int y\ f_X(g^{-1}(y)) \Big/ \left|\frac{dy}{dx}\right|\ dy \\ &= \int g(x)\ f_X(g^{-1}(g(x))) \Big/ \left|\frac{dg(x)}{dx}\right|\ d(g(x) \\ &= \int g(x)\ f_X(x)\ dx \end{aligned}$$