Lecture 2

Probability spaces

Equally likely outcomes

Properties of probability measures

Examples

Probability spaces:
$(\Omega, \mathcal{F}, \mathcal{P})$

- $\Omega = \{\omega\}$: Is the *outcome space*, made up of all basic outcomes, or <u>basic events</u>, ω, of a (random) experiment.

- $\mathcal{F}$: Is the *sigma* (σ) *field* of <u>events</u>, $A, B, C, \ldots$, or of sets for which we wish to compute probabilities.

- P: Is the *probability measure*, or *probability distribution*, which assigns probabilities $P(A), P(B), P(C), \ldots$ to events $A, B, C, \ldots$ in $\mathcal{F}$.

The three axioms of probability measures

Non-negativity

For <u>all</u> sets (events) $A \subset \Omega$,

$$P\{A\} \;\geq\; 0.$$

Additivity

If A and B are <u>disjoint</u> ($A \cap B = \emptyset$) then

$$P\{A \cup B\} \;=\; P\{A\} + P\{B\}.$$

Total mass one

$$P\{\Omega\} \;=\; 1.$$

Equally likely outcomes

Discrete spaces

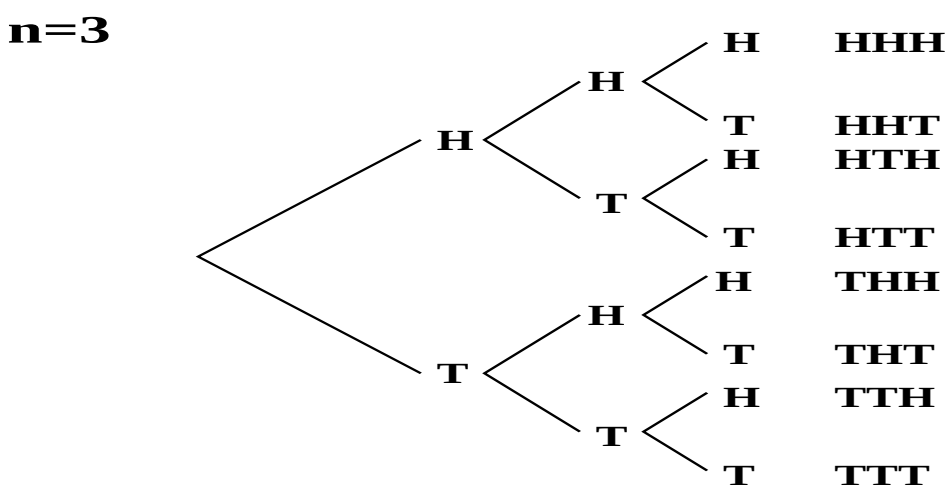
If Ω is <u>finite</u>, and all <u>basic events</u>, ω, (outcomes) in Ω are <u>equally likely</u>, then, for all (complex) events $A \subset \mathcal{F}$,

$$P\{A\} = \frac{\#(A)}{\#(\Omega)}$$

$(\Rightarrow P\{\omega\} = 1/\#(\Omega))$

Example 4: Tossing n fair coins

- $\Omega = \{(i_1, \ldots, i_n)\}, i_j \in \{H, T\}\}$.
- A uniform probability space with $\#(\Omega) = 2^n$.



$$
\begin{aligned}
P\{2 \text{ heads in } 3 \text{ tosses}\} &= P\{(\text{HHT,HTH,THH})\} \\
&= 3/8.
\end{aligned}
$$

Example 5: Proportions

Suppose we have

- $\Omega = \{\text{A collection of individuals}\}$.
- Attributes for individuals
  (Blue eyes, brown eyes, red hair, blond hair, etc)
- The <u>proportion</u> of individuals with attribute A_k is p_k.
  Then

$$p_k = \frac{\#\{\text{individuals with attribute } A_k\}}{\#(\Omega)}$$

We then choose an individual <u>at random</u>. This means

- $P\{\text{The individual has attribute } A_k\} \equiv p_k$

Equally likely outcomes

Continuous spaces

If Ω is a <u>finite</u> subset of $\Re^N$, and all <u>events</u>, A, in Ω have a probability <u>proportional to their size</u>, then,

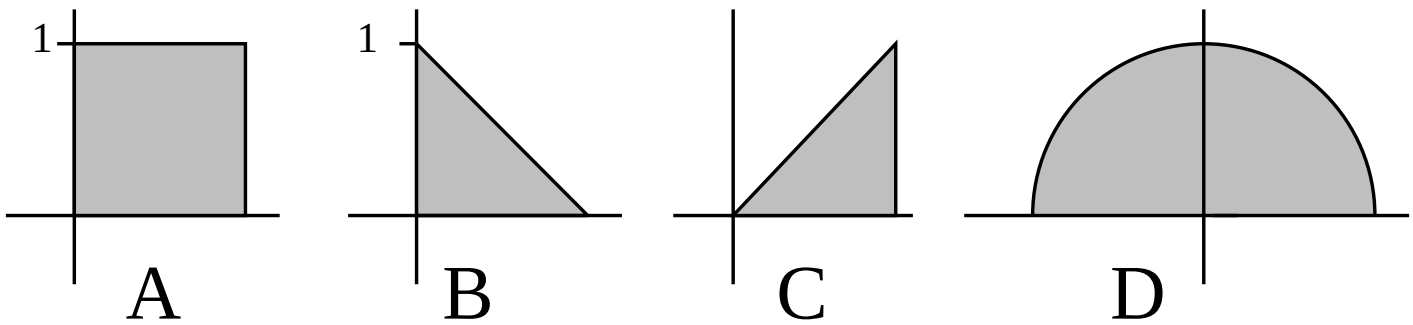
$$P\{A\} \;=\; \frac{|A|}{|\Omega|}$$

where $|A|$ is the "size" of A, measured in accordance with dimension:

- $N = 1: \;\; |A| \;=\;$ length of A
- $N = 2: \;\; |A| \;=\;$ area of A
- $N = 3: \;\; |A| \;=\;$ volume of A
- $N \geq 4: \;$ etc

Probability and area

Drawing a point (X, Y) at random (uniformly) from a set.



Ω	$\lvert\Omega\rvert$	$P\{Y < \frac{1}{2}\}$	$P\{X < Y\}$	$P\{\text{Both}\}$	$P\{\text{Either}\}$
A	1	1/2	1/2	1/8	7/8
B	1/2	3/4	1/2	1/4	1
C	1/2	3/4	0	0	3/4
D	$\pi/2$	?	3/4	?	?+?−?

Properties of P

$$\boxed{P\{B^c\} = P\{\text{does } \underline{\text{not}} \text{ happen}\} = 1 - P\{B\}}$$

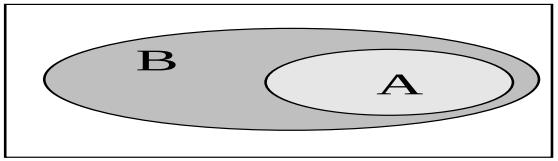
PROOF:

$$1 \;=\; P\{\Omega\} \;=\; P\{B \cup B^c\} \;=\; P\{B\} + P\{B^c\}$$

$$\boxed{A \subset B \Rightarrow P\{B \setminus A\} = P\{B\} - P\{A\}}$$

$$B \setminus A \equiv B \cap A^c \equiv \{B \text{ happens, but } A \text{ does not}\}.$$

PROOF:



$$\begin{aligned} P\{B\} &= P\{(B \cap A) \cup (B \cap A^c)\} \\ &= P\{B \cap A\} + P\{B \cap A^c\} \\ &= P\{A\} + P\{B \cap A^c\} \end{aligned}$$

$$\boxed{A \subset B \Rightarrow P\{B \setminus A\} = P\{B\} - P\{A\}}$$

implies

$$\boxed{A \subset B \Rightarrow P\{A\} \leq P\{B\}}$$

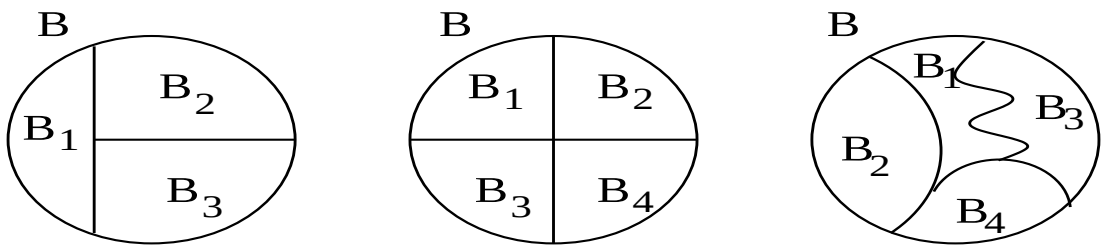
PROOF:

$$P\{B\} \;=\; P\{A\} \;+\; P\{B \setminus A\}$$

Partitions

- $\{B_1, B_2, \ldots\}$ is a partition of $B \subset \Omega$ iff

$$B = \bigcup_k B_k \qquad \text{and} \qquad B_i \cap B_j = \emptyset \ \text{ if } i \neq j.$$



- Dice toss: $B = \Omega$, $B_1 = \{\text{Even number shows}\}, B_2 = \{\text{Odd number shows}\}$.
- Where is the BMW? $B =$ "The game",
  $B_1 = \{\text{I win the BMW}\}$, $B_2 = \{\text{I don't win the BMW}\}$

General (σ) additivity

If $\{B_k\}_{k=1}^n$ is a partition of B, then

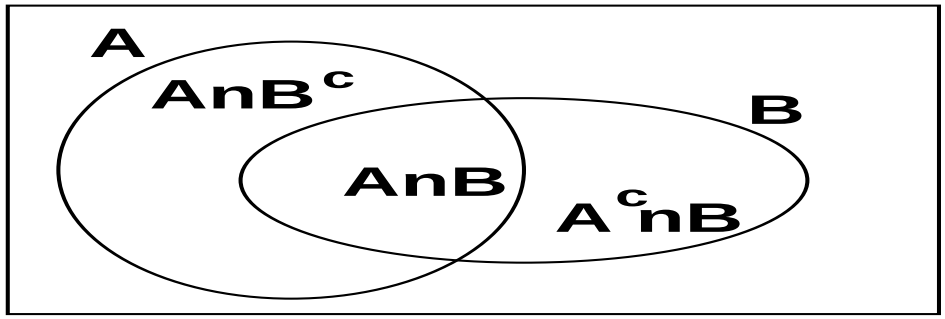
$$P\{B\} \;=\; \sum_{k=1}^{n} P\{B_k\}.$$

PROOF. By induction: $n = 2$ is an axiom. In general

$$\begin{aligned} P\{B\} = P\{\bigcup_{k=1}^{n} B_k\} \;&=\; P\{B_1 \cup \left(\bigcup_{k=2}^{n} B_k\right)\} \\ &=\; P\{B_1\} \;+\; P\{\bigcup_{k=2}^{n} B_k\} \\ &=\; \sum_{k=1}^{n} P\{B_k\}. \end{aligned}$$

- NOTE: The proof also works for $n = \infty$!

$$P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\}$$



$$\begin{aligned}
P\{A \cup B\} &= P\{(A \cap B^c) \cup (B \cap A^c) \cup (A \cap B)\} \\
&= P\{A \cap B^c\} + P\{B \cap A^c)\} + P\{A \cap B\} \\
&= P\{A \setminus (A \cap B)\} + P\{B \setminus (A \cap B))\} + P\{A \cap B\} \\
&= (P\{A\} - P\{B \cap A\}) \ + \ (P\{B\} - P\{B \cap A\}) \\
&\qquad\qquad + P\{B \cap A\} \\
&= P\{A\} + P\{B\} - P\{A \cap B\}
\end{aligned}$$

Examples

- 65% of the population is male
- 60% of the population owns a car
- 80% of car owners are male

- What is the probability that an individual chosen at random EITHER a female OR a pedestrian?

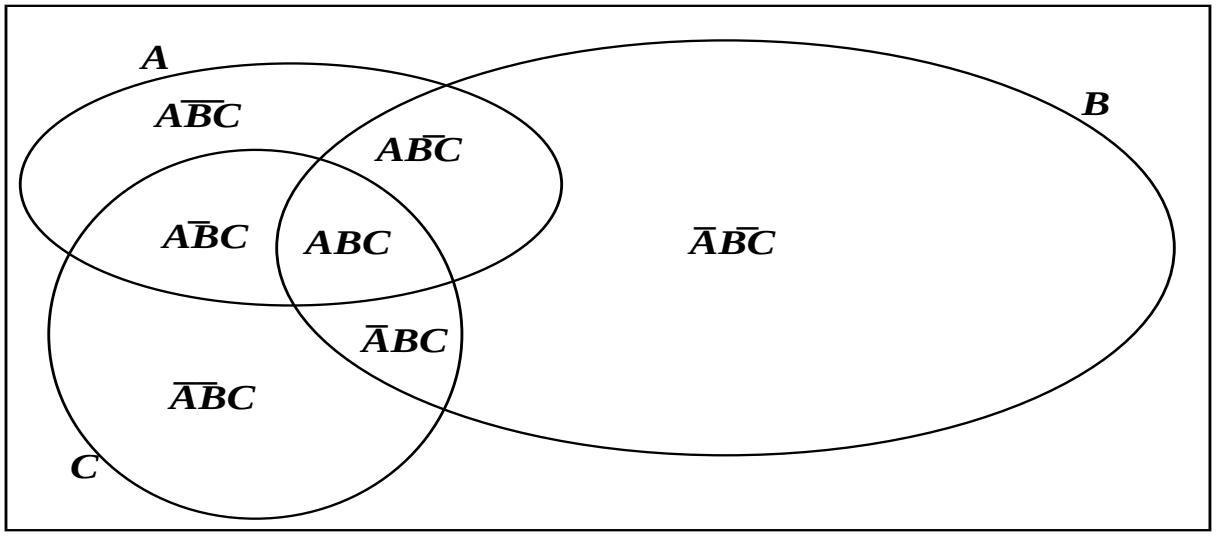
- $P\{\text{Male}\} = 0.65 \quad \Rightarrow \quad P\{\text{Female}\} = 0.35$
- $P\{\text{Car}\} = 0.6 \quad \Rightarrow \quad P\{\text{Pedestrian}\} = 0.4$
- $P\{\text{Car and Male}\} = .8 \times 0.6 = 0.48$

$$
\begin{aligned}
P\{\text{Car AND female}\} &= 0.60 - 0.48 = 0.12 \\
P\{\text{Female AND pedestrian}\} &= 0.35 - 0.12 = 0.23 \\
P\{\text{Female OR pedestrian}\} &= 0.35 + 0.4 - 0.23 = 0.52
\end{aligned}
$$

$$P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\}$$

- This extends (exercise) to

$$\begin{aligned} P\{A \cup B \cup C\} &= P\{A\} + P\{B\} + P\{C\} \\ &\quad - P\{A \cap B\} - P\{B \cap C\} - P\{A \cap C\} \\ &\quad + P\{A \cap B \cap C\} \end{aligned}$$



Inclusion-exclusion

$$P\{\bigcup_{k=1}^{n} A_k\} = \sum_{k=1}^{n} P\{A_k\} - \sum_{j,k=1}^{n} P\{A_j A_k\}$$
$$+ \sum_{j,k,l=1}^{n} P\{A_j A_k A_l\} + \ldots + (-1)^{n-1} P\{A_1 \ldots A_n\}$$

Proof – by induction

$$\begin{aligned} P\{\bigcup_{k=1}^{n} A_k\} &= P\{A_1 \cup \bigcup_{k=2}^{n} A_k\} \\ &= P\{A_1\} + P\{\bigcup_{k=2}^{n} A_k\} - P\{A_1 \cap \bigcup_{k=2}^{n} A_k\} \\ &= P\{A_1\} + P\{\bigcup_{k=2}^{n} A_k\} - P\{\bigcup_{k=2}^{n} (A_1 \cap A_k)\} \\ &= \ldots \end{aligned}$$

Boole's inequality For <u>all</u> collections $\{A_k\}_{k=1}^n$,

$$P\left\{\bigcup_{k=1}^{n} A_k\right\} \leq \sum_{k=1}^{n} P\{A_k\}$$

Proof

$$\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k,$$

where

$$\begin{aligned} B_1 &= A_1 \\ B_k &= A_k \setminus \left(\bigcup_{j=1}^{k-1} B_j\right). \end{aligned}$$