# Lecture 4

Law of total probability

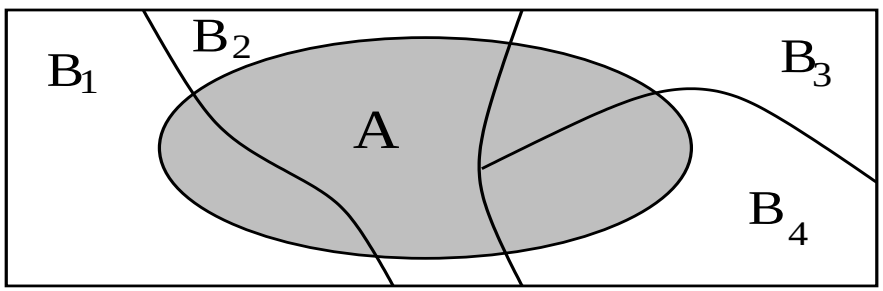
Bayes rule

Sequences of events

# Law of total probability

If $\{B_1, \ldots, B_n\}$ is a <u>partition</u> of $\Omega$, then

$$\boxed{P\{A\} \;=\; \sum_{k=1}^{n} P\{A|B_k\}P\{B_k\}}$$




$$P\{A\} \;=\; P\{A \cap \bigcup_{k=1}^{n} B_k\} \;=\; P\{\bigcup_{k=1}^{n} A \cap B_k\}$$

$$= \sum_{k=1}^{n} P\{A \cap B_k\} = \sum_{k=1}^{n} P\{A|B_k\}P\{B_k\}$$

## Example 1: Colour of second card

- 2 cards are drawn at random from a regular pack of 52. What is the probability that the <u>second</u> card is black?

$$
\begin{aligned}
P\{\text{Second black}\} &= P\{\text{Second black}|\text{First black}\}P\{\text{First black}\} \\
&\quad + P\{\text{Second black}|\text{First red}\}P\{\text{First red}\} \\
&= \frac{25}{51} \times \frac{1}{2} + \frac{26}{51} \times \frac{1}{2} \\
&= \left(\frac{25+26}{51}\right) \times \frac{1}{2} = \frac{1}{2}
\end{aligned}
$$

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By symmetry,

$$P\{\text{Second black}\} = P\{\text{Second red}\}$$

and sum is 1!

## Example 2: Poisonned chocolates

- $k$ chocolates in a box of $N$ are poisoned. 2 chocolates are are drawn at random. What is the probability that the <u>second</u> chocolate is poisoned?

$$
\begin{aligned}
P\{\text{2nd poison}\} &= P\{\text{2nd poison}|\text{1st poison}\}P\{\text{1st poison}\} \\
&\qquad + P\{\text{2nd poison}|\text{1st OK}\}P\{\text{1st OK}\} \\
&= \left(\frac{k-1}{N-1} \times \frac{k}{N}\right) + \left(\frac{k}{N-1} \times \frac{N-k}{N}\right) \\
&= \frac{k(k-1)+k(N-k)}{N(N-1)} \\
&= \frac{k}{N} \\
&= P\{\text{1st poison}\}!!
\end{aligned}
$$

Note: Also symmetry here.

## Example 3: Coins and dice

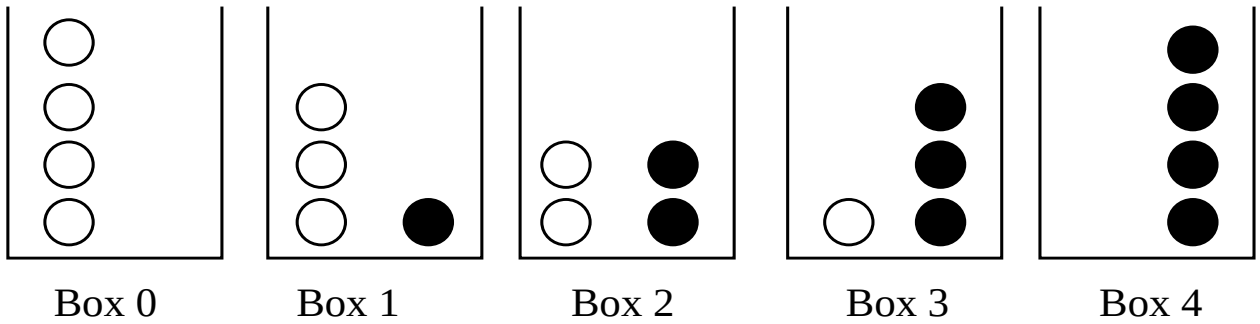
- I toss a biased coin ($P\{H\} = p$) and if I get a H I throw a 6-faced (fair) die, twice. Otherwise, I throw a 4-faced (fair) die, twice. What is the probability that the sum of the two faces is 7?

$$
\begin{aligned}
P\{7\} &= P\{7|\text{ 6-faced die}\}P\{\text{6-faced die}\} \\
&\quad + P\{7|\text{4-faced die}\}P\{\text{4-faced die}\} \\
&= \frac{6}{36} \times p + \frac{2}{16} \times (1-p) \\
&= \frac{1}{8} + \frac{p}{24}
\end{aligned}
$$

Note: $p = 0$ and $p = 1$.

## Example 4: Choosing from boxes

- 5 boxes, with black and white balls as shown. I choose a box at random, and then two balls, <u>without replacement</u>. What is the probability that <u>both</u> are black?

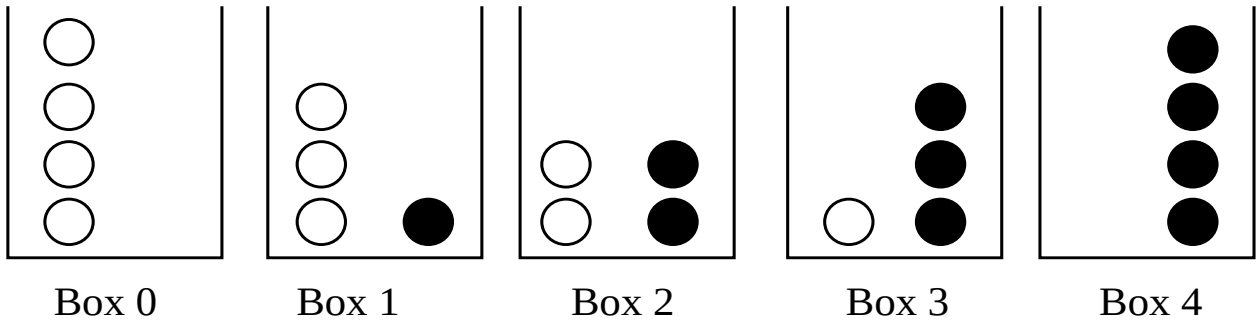



$$
\begin{aligned}
P\{\text{Both black}\} &= \sum_{k=0}^{4} P\{\text{Both black}|\text{Box } k\} \times P\{\text{Box } k\} \\
&= \sum_{k=2}^{4} \frac{k}{4}\frac{k-1}{3}\frac{1}{5} = \frac{1}{5}\left(\frac{2}{4}\frac{1}{3} + \frac{3}{4}\frac{2}{3} + 1\right) = \frac{1}{3}
\end{aligned}
$$

Note: No symmetry!

## Example 4: Another version

- 5 boxes, with black and white balls as shown. I choose a box at random, and then two balls, with replacement. What is the probability that <u>both</u> are black?




$$
\begin{aligned}
P\{\text{Both black}\} &= \sum_{k=0}^{4} P\{\text{Both black}|\text{Box } k\} \times P\{\text{Box } k\} \\
&= \sum_{k=2}^{4} \left(\frac{k}{4}\right)^2 \frac{1}{5} = \frac{1}{5}\left[\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + 1\right] \\
&= \frac{3}{8}
\end{aligned}
$$

## Bayes' Rule

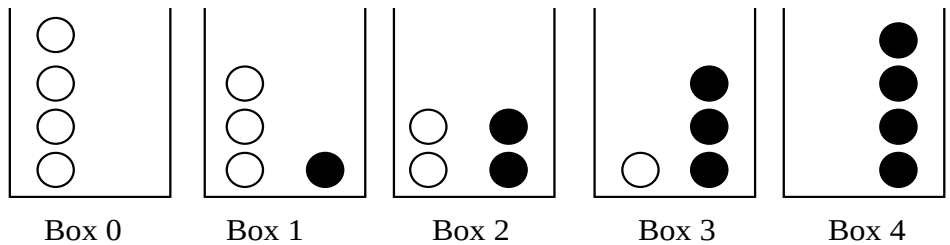
If $A$ is an event, and $\{B_1, \ldots, B_n\}$ a partition of $\Omega$, then

$$P\{B_j|A\} = \frac{P\{A|B_j\}P\{B_j\}}{\sum_{k=1}^{n} P\{A|B_k\}P\{B_k\}}$$

PROOF:

$$\begin{aligned} P\{B_j|A\} &= \frac{P\{B_j \cap A\}}{P\{A\}} \\ &= \frac{P\{A|B_j\}P\{B_j\}}{P\{A\}} \\ &= \frac{P\{A|B_j\}P\{B_j\}}{\sum_{k=1}^{n} P\{A|B_k\}P\{B_k\}} \end{aligned}$$

## Example 4-cont: Choosing from boxes




- Given that both balls were black, what is the probability that I choose Box 2

$$\boxed{P\{B_j|A\} = P\{A|B_j\}P\{B_j\}/\sum_{k=1}^{n} P\{A|B_k\}P\{B_k\}}$$

- $A = \{2 \text{ black balls}\}, \qquad B_j = \{\text{Box } j \text{ chosen}\}$

$$P\{B_2|A\} = \frac{P\{A|B_2\} \times P\{B_2\}}{\sum_{k=1}^{n} P\{A|B_k\}P\{B_k\}} = \frac{\frac{2}{4}\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3}} = \frac{1}{10}$$

$$P\{B_4|A\} = \frac{1 \times \frac{1}{5}}{\frac{1}{3}} = \frac{3}{5} \quad \Rightarrow \quad P\{B_3|A\} = \frac{3}{10}$$

## Example 5: Diagnosis for a random patient

- A blood test can yield the result $P$ or $N$.
- 95% of people with a disease show a $P$ on the test.
- 2% of people without the disease show a $P$ on the test.
- 1% of the population has the disease.
- If I test $P$, what is the probability that I have the disease?

$$P\{P|D\} = 0.95, \quad P\{P|D^c\} = 0.02, \quad P\{D\} = 0.01, \quad P\{D^c\} = 0.99$$

$$\begin{aligned} P\{D|P\} &= \frac{P\{P|D\}\,P\{D\}}{P\{P|D\}\,P\{D\} + P\{P|D^c\}\,P\{D^c\}} \\ &= \frac{(.95)(.01)}{(.95)(.01) + (.02)(.99)} \\ &= \frac{95}{293} \approx 0.32 \end{aligned}$$

## Example 5-cont: Diagnosis for a <u>random</u> patient

$$P\{P|D\} = 0.95,\ P\{P|D^c\} = 0.02,\ P\{D\} = 0.01,\ P\{D^c\} = 0.99$$

$$P\{D|P\} = \frac{(.95)(.01)}{(.95)(.01) + (.02)(.99)} = \frac{95}{293} \approx 0.32$$

$$P\{P|D\} = 0.95,\ P\{P|D^c\} = 0.02,\ P\{D\} = 0.001,\ P\{D^c\} = 0.999$$

$$P\{D|P\} = \frac{(.95)(.001)}{(.95)(.001) + (.02)(.999)} \approx 0.045$$

$$P\{P|D\} = 0.95,\ P\{P|D^c\} = 0.02,\ P\{D\} = 0.1,\ P\{D^c\} = 0.9$$

$$P\{D|P\} = \frac{(.95)(.1)}{(.95)(.1) + (.02)(.9)} \approx 0.84$$

## Sequences of events

- Tossing a coin $n$ times.
- Tossing a coin until a $H$ is seen.
- Choosing 7 balls from one of 8 boxes, with 0–8 white balls per box.

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## Multiplication rule for $n$ events:

$$P\{A_1 A_2 \ldots A_n\}$$
$$= P\{A_1\}P\{A_2|A_1\}P\{A_3|A_1A_2\} \times \ldots P\{A_n|A_1A_2\ldots A_{n-1}\}$$

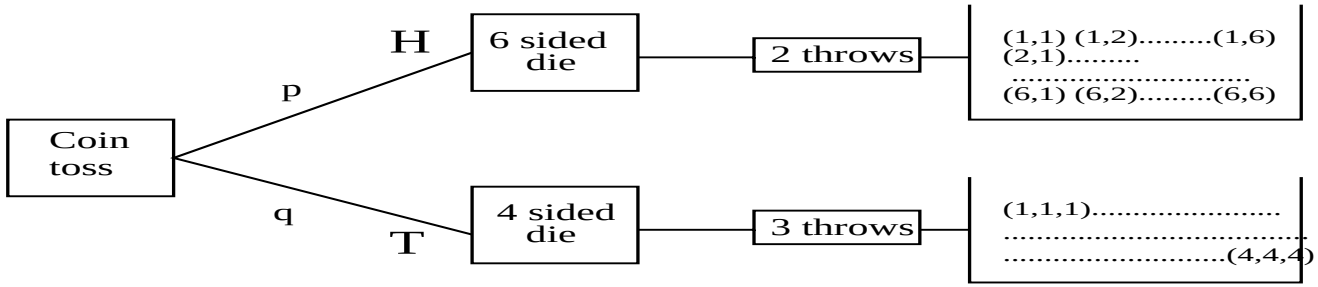
$$\boxed{P\{A_1A_2\ldots A_n\} \;=\; P\{A_1\}P\{A_2|A_1\}\cdots P\{A_n|A_1A_2\ldots A_{n-1}\}}$$

Proof: (By induction)

- For $n = 2$, this is definition of conditional probability.
- For $n \geq 3$

$$\begin{aligned}
&P\{A_1A_2\ldots A_n\} \\
&\quad = P\{(A_1\ldots A_{n-1}) \cap A_n\} \\
&\quad = P\{A_1\ldots A_{n-1}\}\cdot P\{A_n|A_1\ldots A_{n-1}\} \\
&\quad = P\{A_1\}P\{A_2|A_1\}\cdots P\{A_{n-1}|A_1A_2\ldots A_{n-2}\} \\
&\qquad\qquad \times P\{A_n|A_1A_2\ldots A_{n-1}\}
\end{aligned}$$

## Example




- What is the probability that the sum of all faces equals 3? i.e. $P\{(1,2),(2,1),(1,1,1)\}$

$$P\{A_1A_2A_3\} = P\{A_1\}P\{A_2|A_1\}P\{A_3|A_1A_2\}$$

$$P\{A_1A_2A_3A_4\} = P\{A_1\}P\{A_2|A_1\}P\{A_3|A_1A_2\}P\{A_4|A_1A_2A_3\}$$

$$P\{(1,2)\} = P\{H\}P\{1|H\}P\{2|H,1\} = p\tfrac{1}{6}\tfrac{1}{6} = \tfrac{p}{36}$$

$$P\{(2,1)\} = P\{(1,2)\} = \tfrac{p}{36}$$

$$P\{(1,1,1)\} = P\{T\}P\{1|T\}P\{1|T,1\}P\{1|T,1,1\} = q\tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4} = \tfrac{q}{64}$$

$$\Rightarrow P\{\text{Sum is } 3\} = 2\tfrac{p}{36} + \tfrac{q}{64} = \tfrac{1}{6} + p\tfrac{23}{576}$$