Lecture 33

Conditional densities

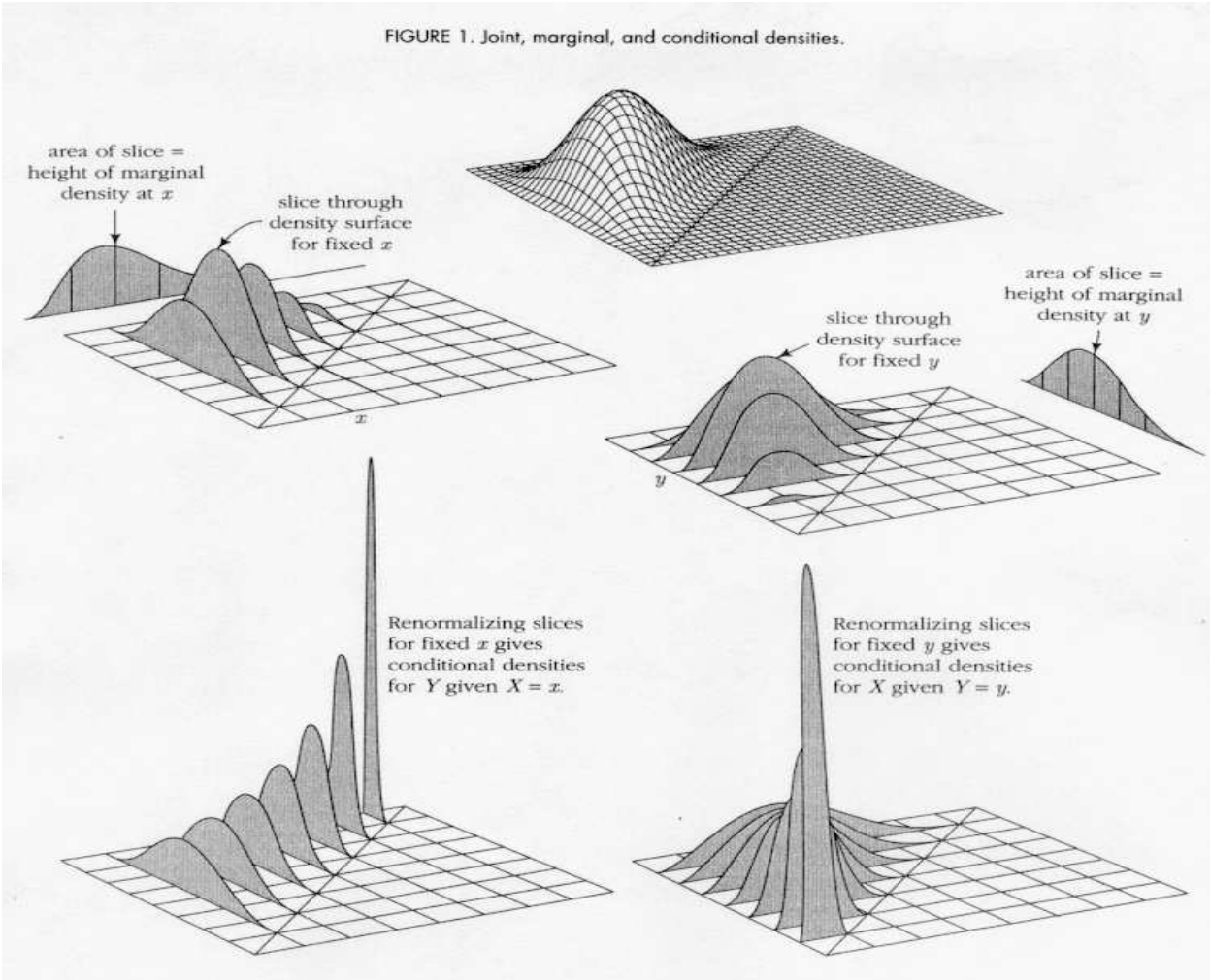
Integral conditioning formula

Continuum Bayes' formula

Conditional expectation

Random sums – again

Conditional: $f_{Y|X}(y|x) \;=\; f_{XY}(x,y) \big/ f_X(x)$



Integral conditioning formula:

$$P\{A\} \;=\; \int_x P\{A|X=x\}\; f_X(x)\; dx.$$

- To simplify the proof, suppose $P\{X \in [a,b]\} = 1$, a, b finite
- Define $a = x_1 < x_2 < \ldots < x_n = b$, $dx_k = x_{k+1} - x_k$.
- By the law of total probability

$$P\{A\} \;=\; \sum_k P\{A|X \in dx_k\}\; P\{X \in dx_k\},$$

- Take a fine partition: i.e. All the dx_k small. Then

$$\begin{aligned}\sum_k P\{A|X \in dx_k\}\; P\{X \in dx_k\} \;&\approx\; \sum_k P\{A|X = x_k\}\; f_X(x_k) dx_k \\ &\approx\; \int_a^b P\{A|X = x\}\; f_X(x)\; dx\end{aligned}$$

Continuum Bayes formula

- In the discrete situation, Bayes' formula gives

$$
\begin{aligned}
P\{X \in dx|A\} &\approx \frac{P\{X \in dx\} \cdot P\{A|X \in dx\}}{P\{A\}} \\
&= \frac{f_X(x) P\{A|X = x\}}{P\{A\}}\, dx
\end{aligned}
$$

- This implies the continuum Bayes' formula

$$
f_{X|A}(x|A) = \frac{f_X(x)\ P\{A|X = x\}}{\int_x P\{A|X = x\}\ f_X(x)\ dx}
$$

Continuous Uniform-Binomial

- Choose a number Π uniformly from [0,1].
- Perform n Bernoulli experiments, with $P\{\text{Success}\} = \Pi$
- Set S_n the number of successes.
- Find distribution of S_n.

$$
\begin{aligned}
P\{S_n = k\} &= \int P\{S_n = k | \Pi = p\} f_\Pi(p)\ dp \\
&= \int_0^1 \binom{n}{k} p^k (1-p)^{n-k}\ dp \\
&= \frac{n!}{k!(n-k)!} \quad \frac{k!(n-k)!}{(k+(n-k)+1)!} \\
&= \frac{1}{n+1}
\end{aligned}
$$

Continuation

- Find conditional distribution of Π given that $S_n = k$.

$$
\begin{aligned}
f_{\Pi|S_n}(p|k) &= \frac{f_\Pi(p)\ P\{S_n = k|\Pi = p\}}{P\{S_n = k\}} \\
&= (n+1)\binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{(k+(n-k)+1)!}{k!(n-k)!}\ p^k (1-p)^{n-k} \\
&= \frac{1}{B(k+1, n-k+1)}\ p^k (1-p)^{n-k}
\end{aligned}
$$

- i.e Π given $S_n = k$ has a Beta$(k+1, n-k+1)$ distribution.

$$f(x) = x^{r-1}(1-x)^{s-1}/B(r,s)$$

Discrete Uniform-Binomial

- $N+1$ boxes, labelled $0, 1, 2, \ldots, N$. Box number b has b black balls and $N-b$ white balls.
- Choose a box at random, and draw n balls, with replacement. S_n is the total number of black balls.

$$
\begin{aligned}
P\{S_n = k\} &= \sum_{b=0}^{n} P\{S_n = k | \text{Box } b \text{ chosen}\} \cdot P\{\text{Box } b \text{ chosen}\} \\
&= \frac{1}{N+1} \sum_{b=0}^{N} \binom{n}{k} \left(\frac{b}{N}\right)^k \left(\frac{N-b}{N}\right)^{n-k} \\
&= \binom{n}{k} \frac{1}{(N+1)N^n} \sum_{b=0}^{N} b^k (N-b)^{n-k}
\end{aligned}
$$

- As $N \to \infty$, this formula converges to the answer for continuous uniform-binomial case. Details: Pitman, p419.

Conditional Expectation

	Discrete	Continuous
$E(Y\|A)$	$\sum_y y\ P\{Y=y\|A\}$	$\int y f_{Y\|A}(y)\ dy$
$E(Y)$	$\sum_k E(Y\|A_k)\ P\{A_k\}$	$\sum_k E(Y\|A_k)\ P\{A_k\}$
$E(Y\|X=x)$	$\sum_y y\ P\{Y=y\|X=x\}$	$\int y f_{Y\|x}(y\|x)\ dy$
$E(Y)$	$\sum_x E(Y\|X=x)$ $\times\ P\{X=x\}$	$\int E(Y\|X=x)$ $\times\ f_X(x)\ dx$
$E(Y)$	$E[E(Y\|X)]$	$E[E(Y\|X)]$
$E(h(Y))$	$E[E(h(Y)\|X)]$	$E[E(h(Y)\|X)]$

Random sums – again

- $X_1, X_2, \ldots$ independent, exponential, parameter λ.
  Density $\lambda e^{-\lambda x}$. MGF $M_X(t) = \lambda/(\lambda - t)$.
- N geometric, parameter p.
  $P\{N = n\} = pq^{n-1}$. MGF $M_N(t) = pe^t/(1 - qe^t)$
- $S_N = X_1 + X_2 + \ldots + X_N$

$$\begin{aligned} M_{S_N}(t) \;=\; E(e^{tS_N}) &= E\left[E(e^{tS_N})\middle|N\right] \\ &= E\left[E\left(e^{tX_1}\cdots e^{tX_N}\right)\middle|N\right] \\ &= E\left[\left(\frac{\lambda}{\lambda - t}\right)^N\right] \end{aligned}$$

- Note: We can now be clever and use the fact that this is just $M_N(s)$, where $e^s = \lambda/(\lambda - t)\ldots$

Cont'n

$$M_{S_N}(t) \;=\; \frac{p\;\lambda/(\lambda-t)}{1-q(\lambda/(\lambda-t)} \;=\; \frac{\lambda p}{\lambda p - t}$$

- More directly:

$$\begin{aligned} E\left(\left(\frac{\lambda}{\lambda-t}\right)^N\right) &= \sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda-t}\right)^n pq^{n-1} \\ &= \frac{p\lambda}{\lambda-t}\sum_{n=1}^{\infty}\left(\frac{\lambda q}{\lambda-t}\right)^{n-1} \\ &= \frac{p\lambda}{\lambda-t}\;\frac{1}{1-\lambda q/(\lambda-t)} \;=\; \frac{\lambda p}{\lambda p - t} \end{aligned}$$

- i.e. $S_N \sim \exp(\lambda p)$
- Note: $E(S_N) \;=\; E(X_1)\cdot E(N) \;=\; \frac{1}{\lambda}\cdot\frac{1}{p}$ was already known, but now we have the full distribution.

Probability generating function

$$\begin{aligned} g_X(s) &= E\left(s^X\right) \\ &= E\left((e^{\ln s})^X\right) \\ &= M_X(\ln s) \end{aligned}$$

$X \sim \text{Bin}(n, p)$

$$M_X(t) = (pe^t + q)^n, \qquad g_X(s) = (ps + q)^n$$

$X \sim \text{Poiss}(\lambda)$

$$M_X(t) = e^{-\lambda(1-e^t)}, \qquad g_X(s) = e^{-\lambda(1-s)}$$

$X \sim \text{Geom}(p)$

$$M_X(t) = \frac{pe^t}{1 - qe^t}, \qquad g_X(s) = \frac{ps}{1 - qs}$$

Properties of the PGF

- X must be a discrete random variable taking values in $\{0, 1, 2, \ldots\}$. Then

$$\begin{aligned} g_X(s) &= E\left(s^X\right) = \sum_{k \geq 0} s^k P\{X = k\} \\ &= P\{X = 0\} + sP\{X = 1\} + s^2 P\{X = 2\} + \ldots \end{aligned}$$

- $g_X(1) = 1, \quad g_X(0) = P\{X = 0\}$
- $g'_X(1) = E[X], \quad g'_X(0) = P\{X = 1\}$
- $\frac{d^n}{ds^n} g_X(s)\Big|_{s=1} = E[X(X-1)(X-2)\cdots(X-n+1)]$
- $\frac{d^n}{ds^n} g_X(s)\Big|_{s=0} = n! P\{X = n\}$

More random sums

- $X_1, X_2, \ldots$ independent, common PGF $g_X(s)$.
- N independent of the X_k, PGF $g_N(s)$.
- $S_N = X_1 + X_2 + \ldots + X_N$

$$
\begin{aligned}
g_{S_N}(s) &= E\left(s^{S_N}\right) \\
&= E\left[E\left(s^{S_N} \middle| N\right)\right] \\
&= E\left[E\left(s^{X_1} \cdots s^{X_N} \middle| N\right)\right] \\
&= E\left[E\left(s^{X_1}\right) \cdots E\left(s^{X_N}\right) \middle| N\right] \\
&= E\left[\left(g_X(s)\right)^N\right] \\
&= g_N\left(g_X(s)\right)
\end{aligned}
$$

$$\boxed{g_{S_N}(s) \;=\; g_N\left(g_X(s)\right)}$$

Example

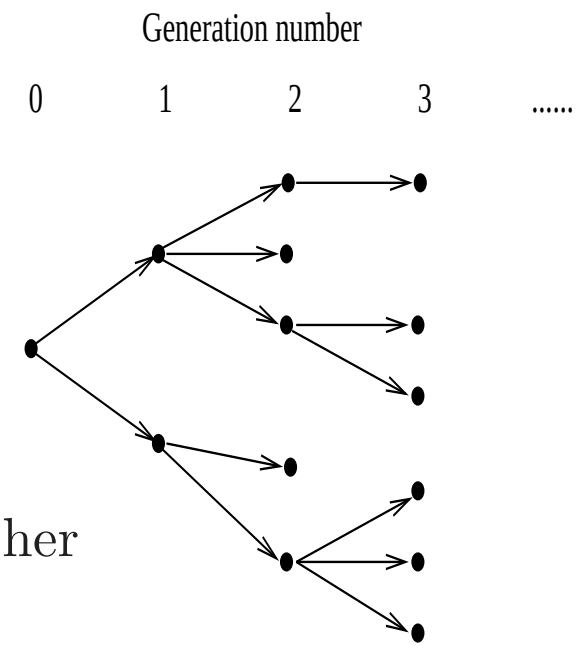
- $X_1, X_2, \ldots$ independent, geometric, parameter p.
  $g_X(s) = \frac{ps}{1-qs}$.

- N Poisson, parameter λ. $\quad g_N(s) = e^{-\lambda(1-s)}$

- $S_N = X_1 + X_2 + \ldots + X_N$

$$\begin{aligned} g_{S_N}(s) &= g_N\left(g_X(s)\right) \\ &= \exp\left(-\lambda(1 - g_X(s))\right) \\ &= \exp\left(-\lambda\left(1 - \frac{ps}{1-qs}\right)\right) \end{aligned}$$

Branching processes



- $X_0 = 1$ individuals in generation 1.
- Individual k in generation n
  has Y_{nk} children, with PGF $g(s)$.
- Family sizes are independent of one another
- X_n individuals in generation n.

$$X_{n+1} = \sum_{k=1}^{X_n} Y_{nk}$$

.

- Find the distribution of X_n.

$$X_{n+1} \;=\; \sum_{k=1}^{X_n} Y_{nk}$$

- Write g_n for the PGF of X_n. Then $g_1 \equiv g$
- Recall $g_{S_N}(s) \;=\; g_N\left(g_X(s)\right)$
- Thus $g_{n+1}(s) \;=\; g_{X_n}\left(g(s)\right)$
- i.e. $g_{n+1}(s) \;=\; g_n\left(g(s)\right)$
- i.e. $g_{n+1}(s) \;=\; g(g(g(\ldots)(s)$
- Finding the mean:

$$\begin{aligned}
E(X_n) \;=\; \frac{d}{ds} g_n(s)\Big|_{s=1} \quad &= \quad \frac{d}{ds} g\big(g(g(\ldots)(s)\Big|_{s=1} \\
&= \quad \Big[\; g'\left(g_{[n-1]}(s)\right) \times g'_{[n-1]}(s)\Big]_{s=1} \\
&= \quad \mu_X \times E(X_{n-1}) \\
&= \quad \mu_X^n
\end{aligned}$$

Conditioning Formulae: Discrete/Continuous summary table

Pitman, pp. 424–425

Handed out in class!