Lecture 20

Square root law

Laws of large numbers

Back to the axioms

Random walk (RW)

Growth rates for RW

Square root law

- $X_1, \ldots, X_n$ uncorrelated, with common mean μ and variance σ^2
- $S_n = X_1 + \ldots + X_n = n\bar{X}_n$

$$
\begin{aligned}
E\{S_n\} &= n\mu, & SD(S_n) &= \sqrt{n}\sigma \\
E\{\bar{X}_n\} &= \mu, & SD(\bar{X}_n) &= \frac{\sigma}{\sqrt{n}}
\end{aligned}
$$

- All of which seems to imply the LAW OF AVERAGES, that sample averages converge to population means or population expectations.

Weak law of large numbers

- $X_1, X_2, \ldots,$ uncorrelated random variables with common mean μ and variance $\sigma^2 < \infty$.
- Note that $\sigma^2 < \infty \;\Rightarrow \mu < \infty$.
- $\bar{X}_n \;=\; \frac{1}{n}\sum_{k=1}^{n} X_k$.
- Then, for any $\varepsilon > 0$,

$$\boxed{P\left\{\left|\bar{X}_n - \mu\right| < \varepsilon\right\} \;\rightarrow\; 1 \qquad \text{as } n \rightarrow \infty}$$

- Proof

$$\begin{aligned} P\left\{\left|\bar{X}_n - \mu\right| \geq \varepsilon\right\} &\leq \left(\frac{SD(\bar{X}_n)}{\varepsilon}\right)^2 \\ &= \frac{\sigma^2}{n\varepsilon^2} \end{aligned}$$

Back to the axioms: Interpretations of probability

Subjective

- What seems reasonable to me. (California)
- What you are told it is. (Old Moscow)
- What we agree on. (Wall Street)

Frequentist

- What happens in the long run:

i.e. We repeat an experiment n times, in an <u>independent</u> fashion, and watch whether the event A occurs or not. We define

$$P(A) = \lim_{n\to\infty}\left(\frac{\text{Number of times } A \text{ occurs in } n \text{ trials}}{n}\right)$$

- Let A be any event belong to $\mathcal{F}$.
- Perform a series of <u>independent</u> experiments, and let X_k be the indicator variable for A on the k-th experiment.

$$X_k = \begin{cases} 1 & \text{if } A \text{ occurs on the } k\text{-th experiment} \\ 0 & \text{otherwise} \end{cases}$$

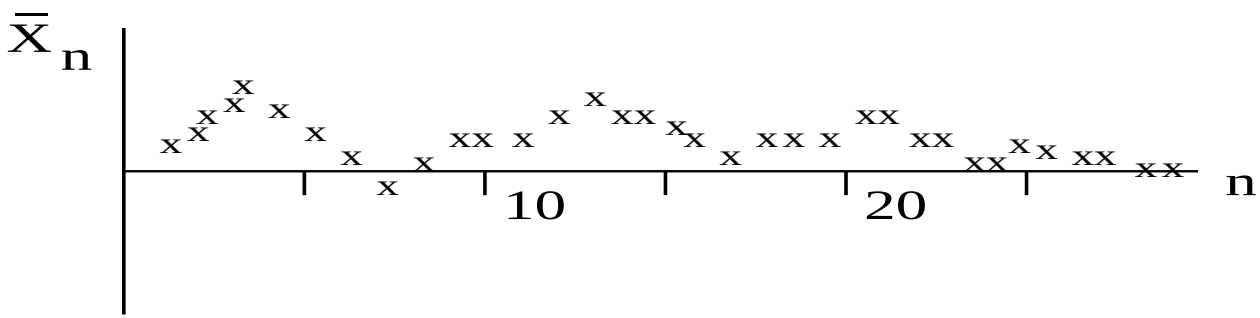
- Note that $E\{X_k\} = P(A)$
- $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ is <u>precisely</u> the above proportion
- By the weak law of large numbers, as $n \to \infty$,

$$P\left\{\left|\frac{\text{Number of times } A \text{ occurs in } n \text{ trials}}{n} - P(A)\right| > \epsilon\right\} \to 0$$

Strong law of large numbers

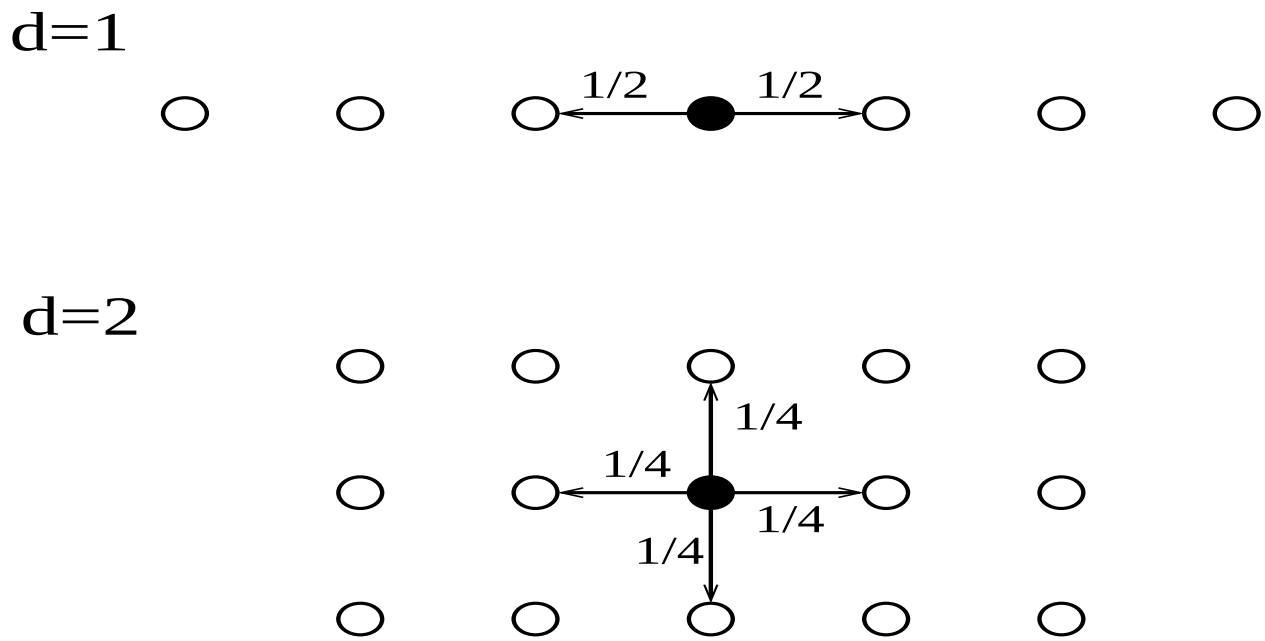
- $X_1, X_2, \ldots,$ independent random variables with common distribution.
- $\Rightarrow$ common mean μ and variance, which are assumed finite.
- $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$.
- Then,

$$P\left\{\lim_{n\to\infty}\left|\bar{X}_n - \mu\right| = 0\right\} = 1$$



PROOF: Hard

Random walk



- Step k represented by X_k, $P\{X_k = \pm 1\} = 1/2$. $(d = 1)$
- Independent steps
- Position after n steps is $S_n = X_1 + \ldots + X_n$.

Growth rates for $d = 1$ random walk

- $S_n = X_1 + \ldots + X_n$
- Take n and k both even, or both odd.

$$
\begin{aligned}
&P\{S_n = k\} \\
&\qquad = \; P\{(n+k)/2 \text{ steps to right, and } (n-k)/2 \text{ steps to left}\} \\
&\qquad = \; \binom{n}{\frac{1}{2}(n-k)} 2^{-n} \qquad\qquad k \in \{-n, -n+2, \ldots, n-2, n\}
\end{aligned}
$$

- NOTE: S_n does NOT have a binomial distribution.
- $E\{X_k\} = 0, \; Var(X_k) = (-1)^2 \frac{1}{2} + (+1)^2 \frac{1}{2} = 1.$
- $\Rightarrow \; Var(S_n) = n, \; SD(S_n) = \sqrt{n}.$

General asymptotics

- Law of large numbers

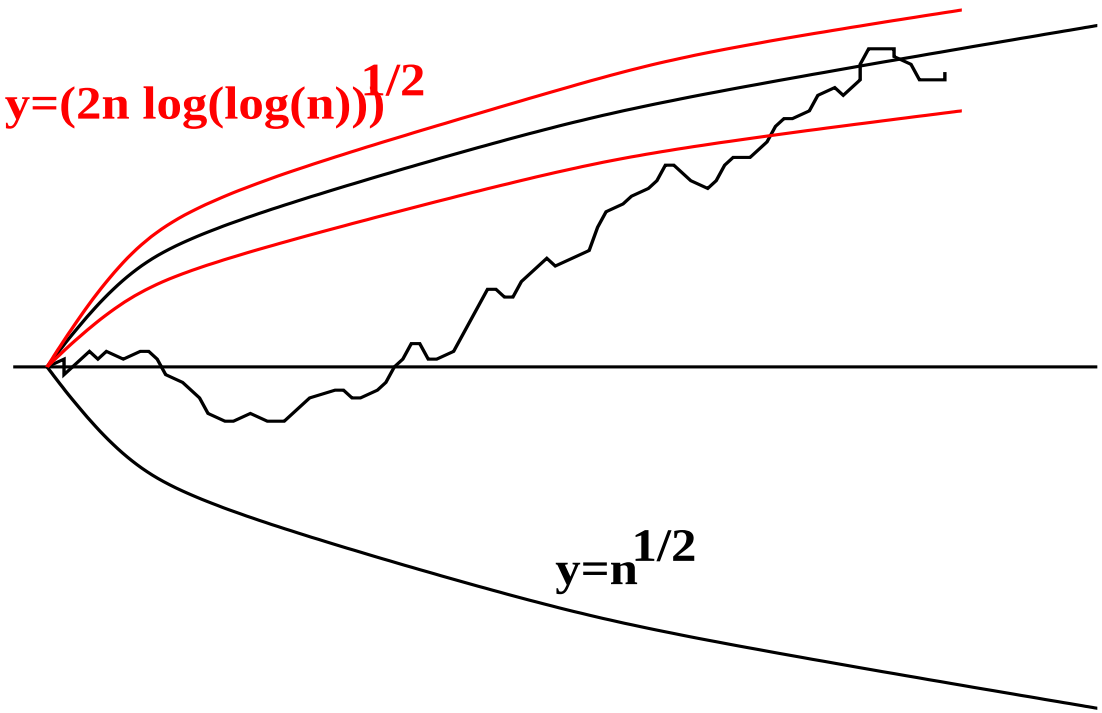
$$P\left\{\left|\frac{S_n}{n}\right| \geq \epsilon\right\} \to 0$$

- Central limit theorem

$$P\left\{x \leq \frac{S_n}{\sqrt{n}} \leq y\right\} \approx \frac{1}{\sqrt{2\pi}} \int_x^y e^{-x^2/2}\, dx$$

- Law of large deviations

$$\begin{aligned} P\{S_n > nx\} &\asymp e^{-n\psi(x)} \\ \text{where} \qquad \psi(x) &= -\log\left(\inf_{t>0}\{e^{-xt} M_{X_1}(t)\}\right) \end{aligned}$$

- Law of the iterated logarithm

$$P\left\{\limsup_{n\to\infty}\frac{S_n}{\sqrt{2n\log(\log(n))}} = 1\right\} = 1$$