Lecture 3

Conditional probability

Gold coins and BMWs

Tree diagrams

Multiplication rule

Independence

Tossing three fair coins

- $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
- $A \equiv \{2 \text{ or more heads}\}$, $\quad P\{A\} = \frac{4}{8} = \frac{1}{2}$
- $B \equiv \{3 \text{ heads}\}$, $\quad P\{B\} = \frac{1}{8} = P\{A \cap B\}$

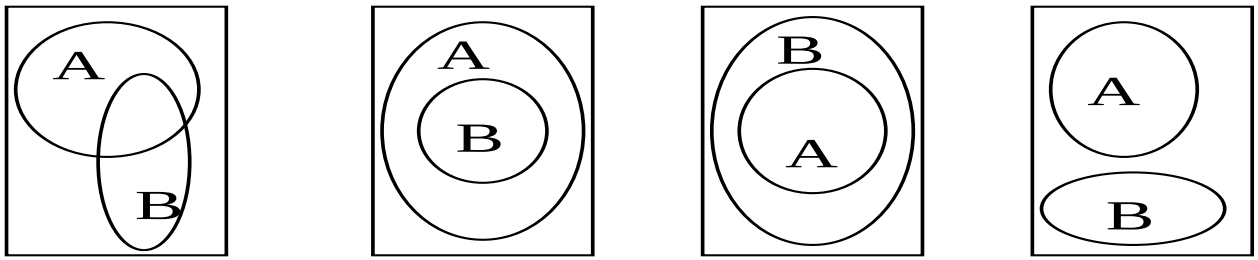
What is the probability of 3 heads, given that at least 2 were seen?

$$
\begin{aligned}
P\{3 \text{ heads}|\text{At least 2 were seen}\} &\equiv P\{B|A\} \\
&= P\{A \cap B|A\} \\
&= \frac{\#\{HHH\}}{\#\{HHH, HHT, HTH, THH\}} \\
&= \frac{1}{4} \\
&= \frac{P\{A \cap B\}}{P\{A\}}
\end{aligned}
$$

Conditional probability:

$$P\{A|B\} \equiv \frac{P\{A \cap B\}}{P\{B\}}$$

- Choosing points at random (uniformly) from a rectangle

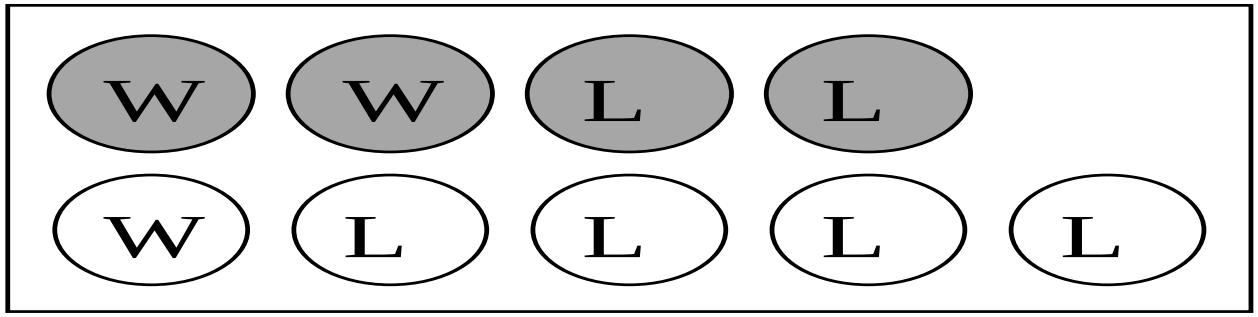


$\sim 2/5$ $\quad$ 1 $\quad$ $\sim 1/2$ $\quad$ 0

In the uniform case, this is equivalent to

$$P\{A|B\} \equiv \frac{\#(A \cap B)}{\#(B)} \quad \text{OR} \quad \frac{|A \cap B|}{|B|}$$

A trivial example



$P\{\text{Win }\} = 3/9 = 1/3$ $\qquad$ $P\{\text{Loose }\} = 6/9 = 2/3$

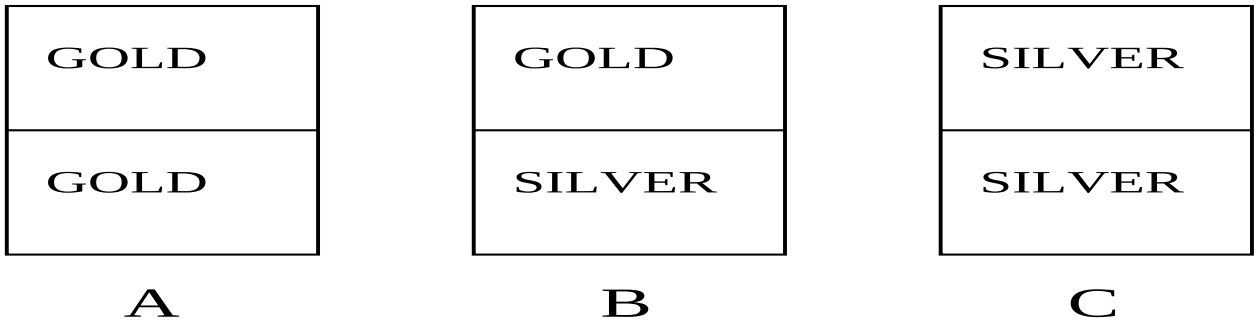
$P\{\text{Grey}\} = 4/9$ $\qquad$ $P\{\text{White }\} = 5/9$

$P\{\text{Win and Grey }\} = 2/9$ $\qquad$ $P\{\text{Win and White }\} = 1/9$

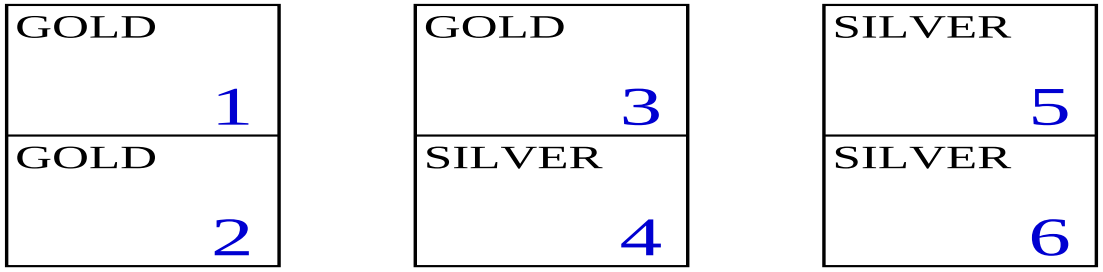
$$P\{\text{Win}|\text{Grey}\} = \frac{P\{\text{Win} \cap \text{Grey}\}}{P\{\text{Grey}\}} = \frac{2/9}{4/9} \left(= \frac{2}{4} = \frac{\#\{\text{Win} \cap \text{Grey}\}}{\#\{\text{Grey}\}} \right)$$

Three problems in probability

Problem # 1: Coins in drawers



- I choose a drawer at random and find a gold coin.
- What is the probability of a <u>second</u> gold, <u>given</u> that one has already been seen?



- $\Omega = \{(1,2),(2,1),(3,4),(4,3),(5,6),(6,5)\}$.
  This is a uniform space.

- $A \equiv \{\underline{\text{First}}\text{ coin is gold}\} \equiv \{(1,2),(2,1),(3,4)\}$
  $B \equiv \{\underline{\text{Second}}\text{ coin is gold}\} \equiv \{(1,2),(2,1),(4,3)\}$
  $A \cap B \equiv \{\text{Both coins are gold} \equiv \{(3,4),(4,3)\}$.

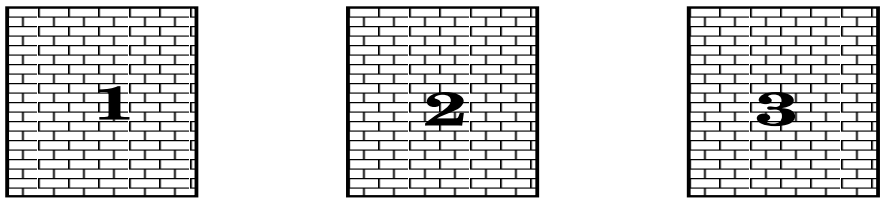
- $P\{A\} = P\{B\} = 3/6 = 1/2, \qquad P\{A \cap B\} = 1/3.$

$$\Rightarrow \; P\{B|A\} \;=\; \frac{P\{A \cap B\}}{P\{A\}} \;=\; \frac{1/3}{1/2} \;=\; \frac{2}{3}$$

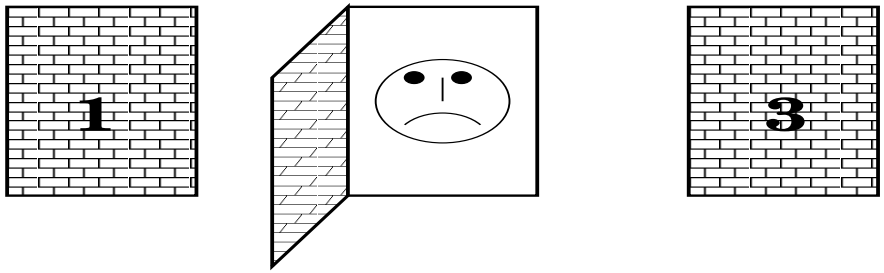
New problem: $\underline{\text{Why}}$ is $P\{B\} = P\{A\}$?

Problem # 2: Where is the BMW?

Stage 1:



Stage 2:



- I select door 1 at Stage 1.
- What is the probability that the BMW is behind door 3 <u>given</u> that I have just been shown that it is not behind door 2?

- To generate a <u>uniform</u> probability space, the compere tosses a coin before he opens a door

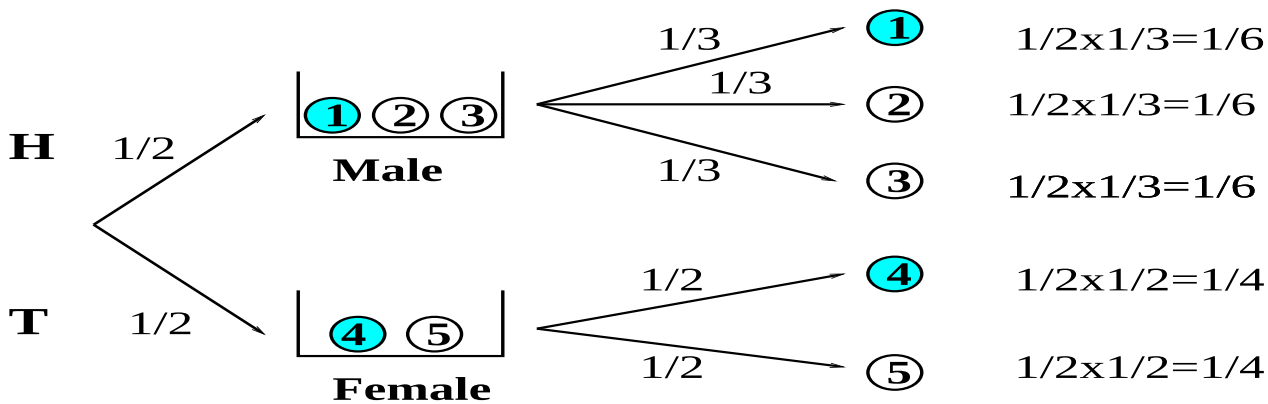
Ω (After I have choosen Door 1)		
Position of BMW		
Door 1	Door 2	Door 3
H$\rightarrow$ 2	H$\rightarrow$ 3	H$\rightarrow$ 2
T$\rightarrow$ 3	T$\rightarrow$ 3	T$\rightarrow$ 2

$$
\begin{aligned}
P\{\text{Door 3}|\text{Door 2 opened}\} &= \frac{P\{\text{Door 3 and Door 2 opened}\}}{\{\text{Door 2 opened}\}} \\
&= \frac{P\{(3, H \rightarrow 2), (3, T \rightarrow 2)\}}{P\{(1, H \rightarrow 2), (3, H \rightarrow 2), (3, T \rightarrow 2)\}} \\
&= \frac{2/6}{3/6} = \frac{2}{3}.
\end{aligned}
$$

Multiplication Rule

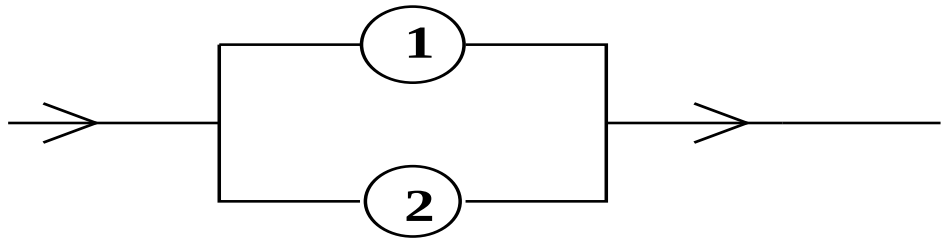
$$P\{A \cap B\} = P\{A|B\}P\{B\}$$

Sequential experiments

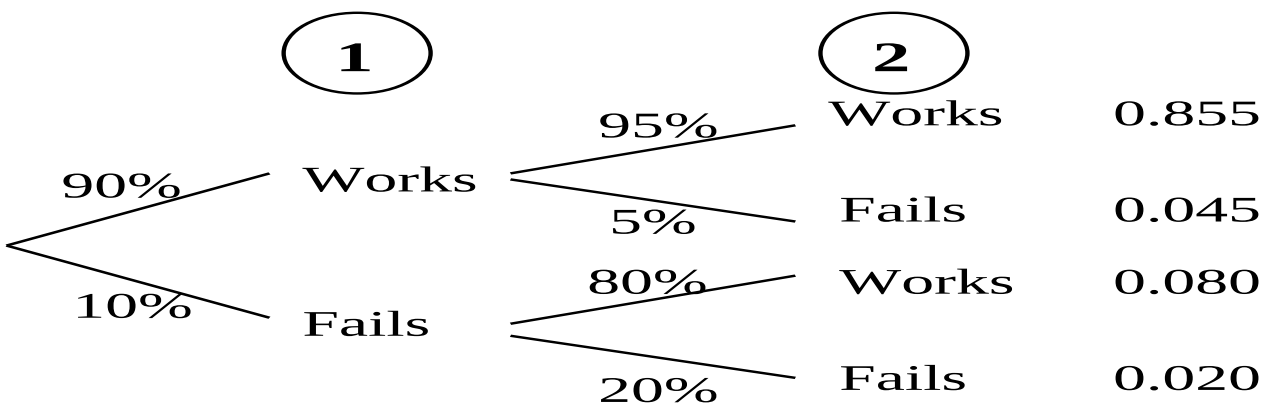


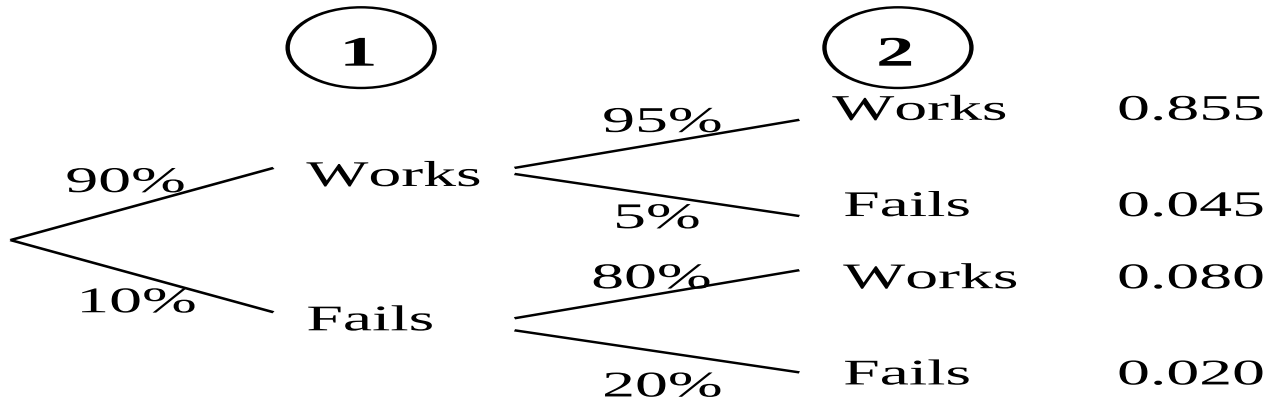
- $P\{1\} = P\{1|\mathrm{H}\} \times P\{\mathrm{H}\} = 1/3 \times 1/2 = 1/6$
- $P\{\text{Male}\} = P\{1 \cup 2 \cup 3\} = 1/6 + 1/6 + 1/6 = 1/2$
- $P\{\text{Green person}\} = P\{1 \cup 4\} = 1/6 + 1/4 = 5/12$

A simple system



- Machine 1 works 90% of the time.
- 95% of the time 1 works, so does 2.
- 80% of the time 1 fails, 2 works.





$$
\begin{aligned}
P\{\text{There is an open path}\} &= P\{1 \cup 2\} \\
&= P\{1\} + P\{2\} - P\{1 \cap 2\} \\
&= 0.9 + (0.855 + 0.080) - 0.855 \\
&= 0.980
\end{aligned}
$$

Next (class) of questions

- If the path is open, what is the probability that 1 has failed?

Independence

A and B are independent $\iff P\{A \cap B\} \equiv P\{A\} \times P\{B\}$

- $A \perp B \iff P\{A|B\} = P\{A|B^c\} = P\{A\}$

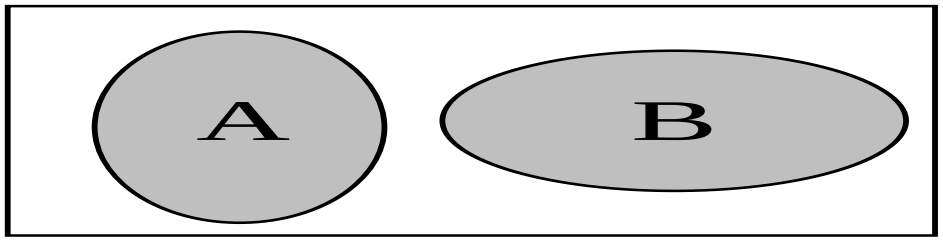
$$(1) \qquad P\{A|B\} = \frac{P\{A \cap B\}}{P\{B\}} = \frac{P\{A\} \times P\{B\}}{P\{B\}} = P\{A\}.$$

$$(2) \quad P\{A|B^c\} = \frac{P\{A \cap B^c\}}{P\{B^c\}} = \frac{P\{A\} - P\{A \cap B\}}{1 - P\{B\}} = \frac{P\{A\}(1 - P\{B\})}{1 - P\{B\}} = P\{A\}.$$

- $A \perp B \iff P\{B|A\} = P\{B|A^c\} = P\{B\}$

Disjoint events are NOT independent

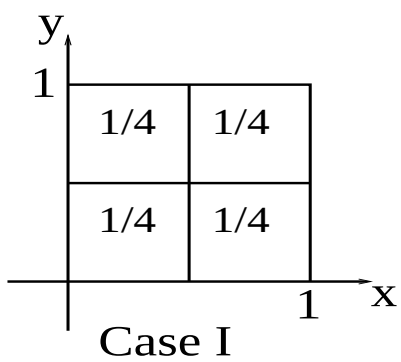
A and B disjoint $\iff A \cap B = \emptyset$

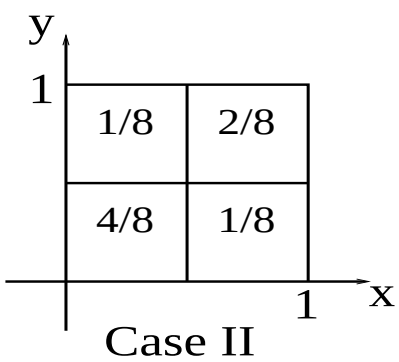


$$P\{A \cap B\} = 0 \neq P\{A\} \cdot P\{B\}$$

(Unless $P\{A\} = 0$ or $P\{B\} = 0$.)

Independence is a property of P



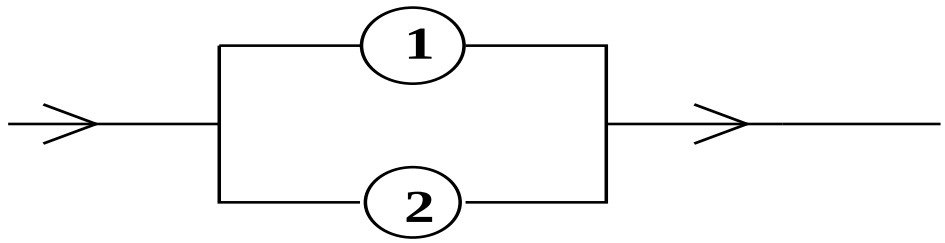


$$A = \{\text{Point is in bottom half of square}\}$$

$$B = \{\text{Point is in left half of square}\}$$

	$P\{A\}$	$P\{B\}$	$P\{A \cap B\}$	$P\{A\|B\}$
Case I	1/2	1/2	1/4	1/2
Case II	5/8	5/8	4/8	4/5

A simple system – with independence

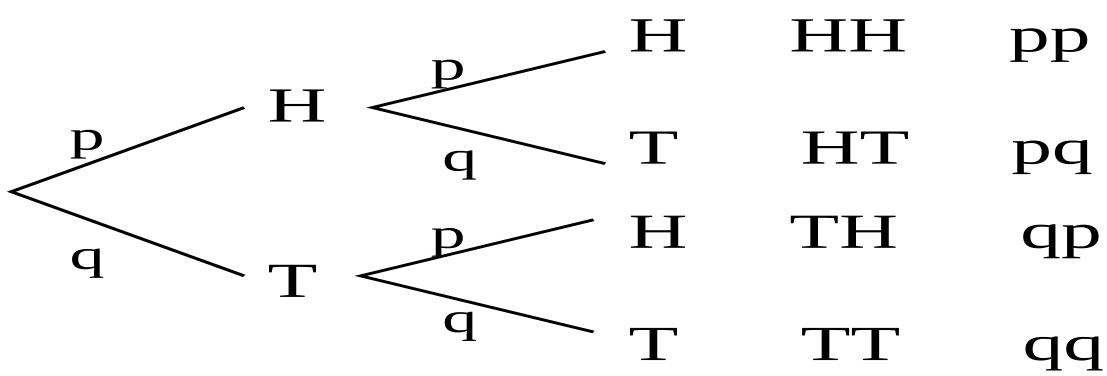


- Machine 1 works 90% of the time.
- Machine 2 works 80% of the time.
- The states of the machines are independent.
- What is the probability that the path is open?

$$
\begin{aligned}
P\{\text{Open path}\} &= P\{\underline{\text{Either}} \text{ machine works}\} \\
&= P\{1 \text{ works}\} + P\{2 \text{ works}\} - P\{\text{Both work}\} \\
&= 0.9 + 0.8 - 0.9 \times 0.8 \\
&= 0.98
\end{aligned}
$$

2 independent biased coins

- $P\{\text{Head}\} = p, \quad P\{\text{Tail}\} = q, \quad (p + q = 1)$



No. of heads	0	1	2
Probability	q^2	$2qp$	p^2