# RANDOM FIELDS AND THEIR GEOMETRY 

Corrections and Commentary

October 7, 2015

Most of the corrections are minor, but some of the original errors have proven annoying to readers. For this we apologise.

However, today, almost a decade after writing the book, we realise that we made a major error in the way we wrote the final chapter, which we called "Non-Gaussian Geometry", and which we started by writing that "This final chapter is ... somewhat of an outlier as far as this book is concerned".

In fact, with the wisdom of hindsight, it is now clear that Theorem 15.9.4 (which we did at least have the good sense to call one of "the main results of this chapter, and indeed of the book") is far more important than we realised at the time. In the meantime it has been given a name, as the Gaussian kinematic formula, (GKF), and this is how Chapter 15 should really have been titled.

At the end of these corrections we have appended a few pages from an Annals of Probability paper, which will help explain the origin of the term GKF. You can read more in the paper itself, or in a far more reader friendly (but also much less rigorous) version of this book in our Saint Flour Lecture Notes, Topological complexity of smooth random functions, available either via Springer or (in an almost final version) on our web sites.

## Chapter 1

$9+15$ : The upper bound in both integrals, as well as the one three lines below, should (obviously) be $+\infty$ rather than $-\infty$.

10+19: $A X$ should be $X A$. Ditto for the second last line of Footnote 9.
10-2: (1.2.7) should read

$$
m_{i \mid j}=m^{i}+\left(x^{j}-m^{j}\right) C_{j j}^{-1} C_{j i} .
$$

10-3: In Footnote 9, the displayed equation should be

$$
A=\left(\begin{array}{cc}
I_{n} & -C_{12} C_{11}^{-1} \\
0 & I_{d-n}
\end{array}\right)
$$

17+8: Replace the second sentence with: Applying (1.3.13) for $u \geq 1$ and checking numerically for $u \in[0,1]$, we have that, for all $u \geq 0$,
18+4: The correct inequality is $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$.
19-14: Equation should be

$$
d\left(\pi_{j_{o}}(t), \pi_{j_{o}}(s)\right) \leq d(s, t)+2 r^{-j_{o}} \leq 3 r^{-j_{o}}
$$

$21+1$ : The upper bound in the integral should be $p^{-1}(\delta)$. The same is true for all integrals on this page.

21-5: $f(t)=f\left(h^{1 / 2}(t)\right.$ should be $f(t)=f\left(W^{1 / 2}(t)\right)$.

22-2: The notation here, and on the next page, is poor. One needs the interpretation that

$$
\sum_{i=1}^{k} h_{i} t_{i}^{\prime} \triangleq \sum_{i=1}^{k} h_{i}\left(0, \ldots, t_{i}^{\prime}, \ldots, 0\right)
$$

with $t_{i}^{\prime}$ appearing in the $i$-th position. This is an element of $\otimes^{k} \mathbb{R}^{N}$, as required.
$23+5$ : The $\widehat{h}_{i}$ under the limit and at the end of the sentence should be $\widehat{h}_{1}$.
23-6: The left hand side of (1.4.10) should be

$$
\mathbb{E}\left\{\left[\eta_{1}^{-k} F\left(t, \eta_{1} t^{\prime}\right)-\eta_{2}^{-k} F\left(s, \eta_{2} s^{\prime}\right)\right]^{2}\right\}
$$

$24+3$ : In the $\eta \neq 0$ part of the definition of $\widehat{F}, F\left(t, \eta t^{\prime}\right)$ needs to be multiplied by $\eta^{-k}$.
$24+8$ : continuity of $f \rightarrow$ continuity of $\widehat{F}$
27-9: $2 \geq a>b$ should be $2 \geq a \geq b$.
33-7: $A_{\delta}$ should be $A^{(n)}$.
$42+19: \pi_{j(t)}$ should be $\pi_{j}(t)$.
44-15: Throughout Lemma 1.5.2 replace log by $\ln$.
45-1: Replace $D_{m}$ in the integral by $D_{\mu}$.
$48+1$ : The proof here is incomplete. Theorem 1.4.1 only establishes that if the entropy (not 'energy' as written) integral $\int_{\delta}^{\infty} p\left(e^{-u^{2} / 2}\right) d u$ is finite, then the entropy integral (1.5.16) converges. In fact, the converse is false. (i.e. There exist processes for which the entropy integral converges, but for which $\int_{\delta}^{\infty} p\left(e^{-u^{2} / 2}\right) d u$ diverges.) Thus the proof only works in one direction.

For the other direction, argue as follows: Assuming that the left hand inequality in (1.5.18) holds for some $\alpha_{1}>0$, show that $M(\varepsilon)$ (defined in the middle of page 47) satisfies

$$
M(\varepsilon)=O\left(e^{N \varepsilon^{-2 /\left(1+\alpha_{1}\right)}}\right)
$$

Then use the relationship between $M$ and $N$ (also in the middle of the page) to show that the entropy integral diverges, and apply Theorem 1.5.4 to complete the proof.

## Chapter 2

$51+3$ : Change $\mathbb{E}\{\|X\|\}$ to $\mathbb{E}\{\|f\|\}$.
52-3: Here and in following three displays $\sqrt{2 \pi}$ should be $\sqrt{\pi / 2}$.
53-2: The assumption that the second derivatives of $f$ and $g$ are bounded is also necessary.
$54+12$ : The minus sign before the integral should be removed here and in the following line.
54-6: The assumption that $h$ is bounded is also necessary.
55-11: Add the assumption that $h$ is bounded.
55-11: $f$ should be $h$. Also for 55-5 and 55-4.
55-4: Change to: To remove the $C^{2}$ and boundedness assumptions, take a sequence of bounded, $C^{2}$ approximations to $h \ldots$.
$56+3$ : There is an implicit assumption in this paragraph that $f_{1}, \ldots, f_{k}$ are independent, standard Gaussians.

56+5: ... from Lemma 2.1.6.
$56+12$ : Change from "trivially" as follows: ... trivially Lipshitz. To compute the Lipshitz constant, note that ...
$57+5$ : There is a $1 / 2$ missing in the exponent. Same problem 3 lines later.
60-11: $\left(X_{1},{ }^{\prime \prime} \ldots, X_{k}^{\prime}\right)$ should be $\left(X_{1}^{\prime \prime}, \ldots, X_{k}^{\prime \prime}\right)$.
61-5: $\sigma_{i j}^{Y}=\mathbb{E}\left\{\tilde{X}_{i}, \widetilde{X}_{j}\right\}$ should be $\sigma_{i j}^{Y}=\mathbb{E}\left\{\tilde{Y}_{i}, \widetilde{Y}_{j}\right\}$.
$63+2$ : In the first sum, $\left(\sigma_{i j}^{Y}-\sigma_{i j}^{X}\right)$ should be $\left(\sigma_{i i}^{Y}-\sigma_{i i}^{X}\right)$.

## Chapter 3

68-14: Delete $\Theta$ and the corresponding brackets from the first term.
72-7: Equals sign missing in $\lambda \psi(t)=\int_{0}^{1} \min (s, t) \psi(s) d s=\ldots$

## Chapter 4

82-3: The right hand bound here is not needed, and, while correct, is misleading. Go directly from $82-4$ to $83+1$.

83-12: A strange typo has appeared here. Throughout the proof, replace $\mathcal{P}_{\uparrow}$ by $\mathcal{P}_{\ell}$ and $\mathcal{P}_{\uparrow+\infty}$ by $\mathcal{P}_{\ell+1}$.

91-3: The last exponent on the RHS should be $2 N / \beta$.
$95+14$ : There is a factor of $T^{N}$ missing on the RHS of (4.4.2).
$96+6$ : The conditioning event should be $f\left(t_{k}\right)=x($ not $=u)$.
98-5: The last line of (4.6.4) is missing the term $\mathbb{P}\left\{M_{u}^{E}\left(M^{\circ}\right) \geq 1\right\}$, with which the iteration begins.

## Chapter 5

$110+13:\|f\| \in L^{2}(\nu)$ should be $f \in L^{2}(\nu)$.
$111+14$ : Replace (5.4.9) and the remainder of the sentence by $W(A)=\Theta^{-1} \mathbb{1}_{A}$, where $A$ is a bounded Borel subset of $\mathbb{R}^{N}$.
$113+6$ : The right hand side needs to be divided by $\prod_{i=1}^{k} h_{i}$.
$114+1: \quad " \beta=\gamma=\delta=0, \alpha=1$ " should be " $\alpha=\gamma=\delta=0, \beta=1$ ".
$114+7: f_{j k}(t)$ should be $f_{k l}(t)$ and $i, j, k$ in the following line should be $i, l, k$.
117+7: Add "up to a multiplicative constant" after "to yield", and after the display replace "absorbing $s_{N-2}$ " by "absorbing all constants".
iv

## Chapter 6

133-2: Should be $F(t, \tau) \in B$.
$137+3$ : The equation should read $f(t)-u=f_{i_{1}}(t)=\cdots=f_{i_{n-1}}(t)=0$, for all subsets $\left(i_{1}, \ldots, i_{N-1}\right)$ of $(1, \ldots, N)$.
$145+3$ : Should be $G_{N}=O(N) \times \mathbb{R}^{N}$
$145+5$ : Should be "... normalized to be the product of Lebesgue measure ...."
$146+20$ : The claim about the non-negativity of the $c_{j}$ is only true if $\psi$ is monotone increasing.

## Chapter 7

162+13: "linear ... in $Y$ " should be "linear ... in $X$ ".
163-1: Should be either $\nabla_{E_{i}} E_{j}=\sum_{k} \Gamma_{i j}^{k} E_{k}$ or $\left(\nabla_{E_{i}} E_{j}, E_{k}\right)=\Gamma_{i j}^{k}$. Both are correct.

## Chapter 8

187-1: $(-1)^{N-j}$ should be $(-1)^{j}$.

## Chapter 9

211-9: (9.3.3) should be (9.3.4).
211-6: Display should read

$$
\mu_{i}(J) \triangleq \#\left\{t \in J: \nabla f_{\mid J}(t)=0, \iota_{-f, J}(t)=i, f(t) \geq u, \nabla f(t) \in N_{t} I^{N}\right\}
$$

211-2: Delete "working with the definition of the $C_{i}$ ".

## Chapter 10

$222+1: \rho \leq$ should be $\rho<$.
$222+3: t \in M$ needs to be added to the definition of the region.
$238+1$ : There is nothing wrong here, but the discussion could have been clearer. In particular, despite the last paragraph on p237, (10.5.4) is still in the most general setting $M \subset$ $\widetilde{M} \subset \mathbb{R}^{\ell}$. On the other hand, (10.5.5) relates to the situation $M \subset \widetilde{M} \equiv \mathbb{R}^{\ell}$, and so $\operatorname{dim} \widetilde{M} \triangleq N \equiv \ell$, which is why $N$ disappears in (10.5.5). (It would have been more natural to write (10.5.5) with $\ell$ replaced by $N$, and it should probably be read that way.)

246-13: $l=k-j=0$ should be $m=j-i=0$.

249-7: $-\widetilde{R}+\kappa I^{2} / 2$ should be $-\left(\widetilde{R}+\kappa I^{2} / 2\right)$.

## Chapter 11

265-2: whether $\lambda_{2}$ is finite, or not...
267-12: $D=N(N+1) / 2+K$ should be replaced by $D=N^{2}+K$. Similarly, the integration over $\mathbb{R}^{N(N+1) / 2}$ in (11.2.11) should be replaced by integration over $\mathbb{R}^{N^{2}}$. The same in true in (11.2.13), the three related integrals on page 272, and the one on page 275. Similarly, the $N(N+1) / 2$ in the first line of the footnote should be $N^{2}$.
$278+14$ : Condition $(\mathrm{g})$ is not needed for Lemma 11.2.11. The proof requires no real changes. At 278-10, replace "it follows from (11.2.2)" by "it follows from the continuity of the $f_{j}^{i}$ ". At 279-9, replace "By assumption (g)" by "By assumption (c)".
This condition, and others like it, are needed only for proving the expectation metatheorem, Theorem 11.2.1.

279-2: Condition (g) is not needed for Lemma 11.2.12.
280-6: The condition that all second derivatives have finite variance is not required. It appears nowhere in the proofs.

This condition, and others like it, are needed only for proving the expectation metatheorem, Theorem 11.2.1.
$281+1$ : Condition (d) is not required.
$281+11$ : Add "for some $\alpha>0$ " immediately after (11.3.1).
$281+18$ : Theorem 11.3.3 should read "Under the conditions of Theorem 11.3.1.
$282+22$ : In (a) the condition $\mathbb{E}\left\{(X Y f)^{2}\right\}<\infty$ is not required.
282-15: Condition (d) is not required.
290-7: $Q$ must be a positive definite square root of $\Lambda^{-1}$.
$299+1$ : Equation (11.8.1) holds in wide generality, once all the terms are properly defined. However, in the notation of Chapter 11, it only makes sense for isotropic fields with variance and second spectral moment both equal to one.

## Chapter 12

$307+4$ : Delete the (meaningless, since $X Y$ is not in the right tangent space) second line of the equation.

310+9: Replace $x I$ by $x g(X, Y)$ here and in the following line.

Chapter 13

333-11: The power of $\lambda$ in (13.2.1) should be $l / 2$.

## Chapter 14

$352+17$ : In Condition (iii) delete "For all $t \in M$ ".
$357+8$ : The term $P_{L} \nabla h(t)$ in the numerator should be $P_{L}^{\perp} \nabla h(t)$
$376+5$ : Four (not two) applications of l'Hôpital's rule are needed.
$377+9$ : Should be $\left(\Psi_{*}\left(X_{t}\right)\right) f=X_{t} f$.
383-10: On the other hand, if $\pi \leq T \leq 2 \pi, \ldots$
385-19: $M_{N}$ has not been defined. It is $\sup _{t \in T} f(t)$, where $f$ is the cosine field on $\mathbb{R}^{N}$.

## Chapter 15

394+1: The $\widetilde{y}$ that appears here is the expansion of $y$ over an ambient manifold $\widetilde{M}$, needed for Morse theory to be applied.

398-1: All four appearences of $n / 2-1$ should be $(n-1) / 2$.
404-3: $G_{n, n^{-1}}$ should be $G_{n, \sqrt{n}}$ and $\nu_{n, n^{-1}}$ should be $\nu_{n, \sqrt{n}}$.
$418+10:$ Delete the term $g_{\mathbb{R}^{k}}$.
418-13: The expression $\mathcal{L}_{n-1-i}^{1 / n}\left(\pi_{\sqrt{n}, n, k}^{-1} D, \widetilde{D}_{n-k-1+j}\right)$ is a little misleading. Technically, for this to be consistent with previous notation, such as at (10.7.1), it would be necessary for $\widetilde{D}_{n-k-1+j}$ to be a subset of $\pi_{\sqrt{n}, n, k}^{-1} D$ which, in view of the definition of $\widetilde{D}_{n-k-1+j}$ five lines above, is only 'morally' true, in terms of isometric embeddings. The notation should therefore be thought of in terms of the warped metric This should be kept in mind in all further uses of this curvature.
418-2: The second expression should read $\widetilde{\nabla}_{F_{1}}^{\sigma} F_{2}=\nabla_{F_{1}}^{2} F_{2}-\sigma^{2} g_{2}\left(F_{1}, F_{2}\right) \nabla \sigma^{2}$.
$420+12$ : Display should read $\nu_{k-j}=\sum_{r=1}^{n-k-1} F_{r}$.
$421+12: \quad \widetilde{D}_{n-k-1+j} \cap S\left(\mathbb{R}^{k}\right)=\emptyset$ should be $D_{j} \cap S_{\sqrt{n}}\left(\mathbb{R}^{k}\right)=\emptyset$ here, and at $423+7$ and $423+9$. The condition is written correctly in the statement of Lemma 15.9.2.
$424+3$ : In order that the $\mathcal{M}_{i}^{\gamma}$ are actually coefficients in the tube formula, we also need the assumptions of Corollary 10.9.6 to hold. Without these, they are as implicitly defined in the last display on the page.
424-6: $(2 \pi)^{-i \hat{A} \frac{1}{2}}$ should be $(2 \pi)^{-i / 2}$.
424-2: The integrability condition (15.9.3) needs to be added to the conditions of Theorem 15.9.4.

426-1: $\gamma_{\mathbb{R}^{l}}$ should be just $\gamma$ (i.e. $\gamma_{\mathbb{R}}$ ) throughout (15.10.6)
$427+5: \gamma_{\mathbb{R}^{k}}$ and $\gamma_{\mathbb{R}^{l}}$ should be simply $\gamma$ here and throughout the proof.
429-9: $e^{-x^{2} / 2}$ should be $e^{-u^{2} / 2}$.

# GAUSSIAN PROCESSES, KINEMATIC FORMULAE AND POINCARÉ'S LIMIT 

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#### Abstract

We consider vector valued, unit variance Gaussian processes defined over stratified manifolds and the geometry of their excursion sets. In particular, we develop an explicit formula for the expectation of all the Lipschitz-Killing curvatures of these sets. Whereas our motivation is primarily probabilistic, with statistical applications in the background, this formula has also an interpretation as a version of the classic kinematic fundamental formula of integral geometry. All of these aspects are developed in the paper.

Particularly novel is the method of proof, which is based on a an approximation to the canonical Gaussian process on the $n$-sphere. The $n \rightarrow \infty$ limit, which gives the final result, is handled via recent extensions of the classic Poincaré limit theorem.


1. Introduction. The central aim of this paper is to describe a new result in the theory of Gaussian related fields, along with some of its implications to both geometry, and to a lesser extent, to statistics.

The basic object of interest is a $\mathbb{R}^{k}$ valued random field $y$ defined on a parameter space $M$ and its excursion sets

$$
\begin{equation*}
A(f, M, D) \triangleq \stackrel{\Delta}{\triangleq}\{t \in M: y(t) \in D\} \tag{1.1}
\end{equation*}
$$

where $D \subset \mathbb{R}^{k}$. For most of the paper, we shall take $M$ and $D$ to be $C^{2}$ stratified manifolds in $\mathbb{R}^{N}$ and $\mathbb{R}^{k}$, respectively.

Stratified manifolds are basically sets that can be partitioned into the disjoint union of manifolds, so that we can write

$$
\begin{equation*}
M=\bigsqcup_{j=0}^{\operatorname{dim} M} \partial_{j} M \tag{1.2}
\end{equation*}
$$

where each stratum, $\partial_{j} M, 0 \leq j \leq \operatorname{dim}(M)$, is itself a disjoint union of a number of $j$-dimensional manifolds. A typical 3-dimensional example is given by the

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Fig. 1. A saggy couch under stress: A stratified manifold with its excursion sets.
saggy couch of Figure 1, in which case $\partial_{3} M$ is the interior of the couch; $\partial_{2} M$ the collection of the six sides, some concave and some convex; $\partial_{1} M$ is made up of the 12 edges; and $\partial_{0} M$ contains the 8 corner vertices. In Figure 1, an excursion set might be the grey area where the stress $y$ is greatest.

Our aim is to study the global geometry of excursion sets, as measured through their Lipschitz-Killing curvatures, $\mathcal{L}_{j}(A(f, M ; D)), j=0, \ldots, \operatorname{dim}(M)$. In particular, since these curvatures are random variables, we shall be interested in computing their expectations. We shall define Lipschitz-Killing curvatures below. However, if you are unfamiliar with them, at this stage it suffices to know that $\mathcal{L}_{N}(A)$ is a measure of the volume of $A, \mathscr{L}_{N-1}(A)$ a measure of its surface area, and $\mathscr{L}_{0}(A)$ its Euler characteristic, an important topological invariant.

We cannot do this for all $y$. For a start, $y$ will both have to be smooth enough for basic differential geometric techniques to be applicable. Thus, a basic requirement will be that $y$ has, with probability one, $C^{2}$ sample paths. Furthermore, writing $y=$ $\left(y_{1}, \ldots, y_{k}\right)$, we shall assume that the $y_{i}$ are independent, identically distributed (hereafter i.i.d.) centered Gaussian processes of constant variance, which we take to be 1 . The processes are, however, not assumed to be stationary. For such a $y$, we shall prove that

$$
\mathbb{E}\left\{\mathcal{L}_{i}(A(y, M, D))\right\}=\sum_{j=0}^{\operatorname{dim} M-i}\left[\begin{array}{c}
i+j  \tag{1.3}\\
j
\end{array}\right](2 \pi)^{-j / 2} \mathcal{L}_{i+j}(M) \mathcal{M}_{j}^{\gamma}(D)
$$

where the combinatorial flag coefficients are defined below at (4.3) and the $\mathcal{M}_{j}^{\gamma}$, described and defined in Sections 3 and 6.1, are certain (Gaussian) Minkowski functionals that to a certain extent, play the role of Lipschitz-Killing curvatures in Gauss space. The Lipschitz-Killing curvatures on both sides of (1.3) are computed with respect to a specific Riemannian metric induced on $M$ by the component processes $y_{j}$. Note, however, that $\mathcal{L}_{0}(A)$ is the Euler-Poincaré characteristic of $A$, and so independent of any Riemannian structure [cf. Theorem 4.1 for a formal statement of (1.3)].
1.1. What is new here? The result (1.3) has a long history. If $M$ is a simple interval $[0, T], y$ is real valued and stationary, and $D=[u, \infty)$, then (1.3) is essentially the famous Rice formula, which gives the mean number of upcrossings of the level $u$ by $f$, and dates back to 1939 [15] and 1945 [16]. Since then, there have been tens, if not hundreds, of papers extending the original Rice formula in many ways, with the developments up until 1980 summarized in [1]. More recently, there was a series of papers by Worsley (e.g., [24-26, 28]) that were important precursors to the general theory presented in this paper. However, the first precursor to (1.3), at the level of processes over manifolds with $C^{2}$ boundaries, appeared only in 2002 in [21], where we considered only the first Lipschitz-Killing curvature $\mathcal{L}_{0}\left(A_{u}(f, M)\right)$ and then only for real valued $y$. In [20], one of us (JET) extended this to vector valued $y$, which allowed for the derivation of the far more general, and far more elegant, (1.3) for the first Lipschitz-Killing curvature.

What is new here then is the extension to parameter spaces as general as stratified manifolds, and the extension to all Lipschitz-Killing curvatures. Both of these are important for applications. However, perhaps more important, and certainly more novel than either of these, is the method of proof. The proofs in the current paper are new, and far more geometric than the earlier ones. In particular, the proof in [20] progressed primarily by evaluating both sides of (1.3) and then showing that they were equivalent. The current proof starts on the left-hand side and, eventually, yields the right-hand side. The geometric nature of the current proof also explains why the two sides should be equal.
1.2. Statistical implications. The general structure of (1.3) has significant implications for a class of problems out of the purely Gaussian scenario. Taking $F: \mathbb{R}^{k} \rightarrow \mathbb{R}$ to be piecewise $C^{2}$, along with appropriate side conditions, and defining a (now non-Gaussian) process

$$
\begin{equation*}
f(t)=F(y(t))=F\left(y_{1}(t), \ldots, y_{k}(t)\right), \tag{1.4}
\end{equation*}
$$

with $y$ Gaussian as above, it follows immediately from (1.3) that

$$
\begin{align*}
& \mathbb{E}\left\{\mathcal{L}_{i}(A(f, M,[u, \infty)))\right\} \\
& \quad=\sum_{j=0}^{\operatorname{dim} M-i}\left[\begin{array}{c}
i+j \\
j
\end{array}\right](2 \pi)^{-j / 2} \mathcal{L}_{i+j}(M) \mathcal{M}_{j}^{\gamma}\left(F^{-1}[u,+\infty)\right) . \tag{1.5}
\end{align*}
$$

Non-Gaussian processes of the form (1.4) appear naturally in a wide variety of statistical applications of smooth random fields (e.g., [2-4, 19, 24-26] with an excellent introductory review in [27]).

An additional and extremely important application of (1.3) lies in the so called "Euler characteristic heuristic" that for a wide range of random fields $f$,

$$
\left|\mathbb{P}\left\{\sup _{t \in M} f(t) \geq u\right\}-\mathbb{E}\left\{\mathscr{L}_{0}(A(f, M,[u, \infty)))\right\}\right| \leq \operatorname{error}(u),
$$

where $\operatorname{error}(u)$ of a smaller order than both of the other terms as $u \rightarrow \infty$. In the Gaussian case, this heuristic is now a well-established theorem, and the error term is known to be of $\operatorname{order} \exp \left(-u^{2}(1+\eta) / 2\right.$ ) (for an identifiable $\eta>0$ ) while both the probability and expectation are of order $\exp \left(-u^{2} / 2\right)$ [22]. The ability to compute the expectation therefore provides useful, explicit approximations for the excursion probability.
1.3. Geometry. One of the basic results of integral geometry is the so-called kinematic fundamental formula (henceforth KFF), which in its simplest form, states that for nice subsets $M_{1}$ and $M_{2}$ of $\mathbb{R}^{n}$,

$$
\begin{align*}
\int_{G_{n}} & \mathcal{L}_{i}\left(M_{1} \cap g_{n} M_{2}\right) d v_{n}\left(g_{n}\right) \\
& =\sum_{j=0}^{n-i}\left[\begin{array}{c}
i+j \\
i
\end{array}\right]\left[\begin{array}{c}
n \\
j
\end{array}\right]^{-1} \mathscr{L}_{i+j}\left(M_{1}\right) \mathcal{L}_{n-j}\left(M_{2}\right) . \tag{1.6}
\end{align*}
$$

Here, $G_{n}$ is the isometry group of $\mathbb{R}^{n}$ with Haar measure $v_{n}$ normalized so that for any $x \in \mathbb{R}^{n}$ and any Borel $A \subset \mathbb{R}^{n}, v_{n}\left(\left\{g_{n} \in G_{n}: g_{n} x \in A\right\}\right)=\mathscr{H}_{n}(A)$, where $\mathscr{H}_{n}$ is $n$-dimensional Hausdorff measure. (See $[12,18]$ for $M_{j}$ elements of the convex ring or similar, and [6] for more esoteric $M_{j}$ closer to the spirit of this paper.)

Now reconsider (1.3). Taking ( $\Omega, \mathcal{F}, \mathbb{P}$ ) as the probability space on which $y$ lives, (1.3) can be rewritten as

$$
\begin{align*}
& \int_{\Omega} \mathcal{L}_{i}\left(M \cap(y(\omega))^{-1} D\right) d \mathbb{P}(\omega)  \tag{1.7}\\
& \quad=\sum_{j=0}^{\operatorname{dim} M-i}\left[\begin{array}{c}
i+j \\
j
\end{array}\right](2 \pi)^{-j / 2} \mathcal{L}_{i+j}(M) \mathcal{M}_{j}^{\gamma}(D) .
\end{align*}
$$

Written this way, it is clear on comparing (1.6) and (1.7) that our main result can now be interpreted as a KFF over Gaussian function space, rather than over the isometry group on Euclidean space. We find this interpretation novel and intriguing bridging as it does between a probabilistic problem and a geometric answer of classic form.
1.4. More on stratified manifolds. Although (1.2) gives the basic structure of a stratified manifold, certain assumptions need to be made for the results of this paper to hold. In particular, we need to assume that each connected component in each stratum is a $C^{2}$ manifold. More importantly, we need to assume that both $M$ and $D$ are Whitney stratified spaces (see $[10,14]$ ) which ensures that the various strata are "glued together" in a smooth way. There are further technical assumptions of "tameness" and "regularity" that can be found in Chapter 15 of [3]. One of these is local convexity; a demand much weaker than convexity (the saggy couch is locally convex), which while not necessary, is something that we shall assume in


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