Central limit theorems for some random simplicial complexes.

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March 2014.

Simplicial complexes

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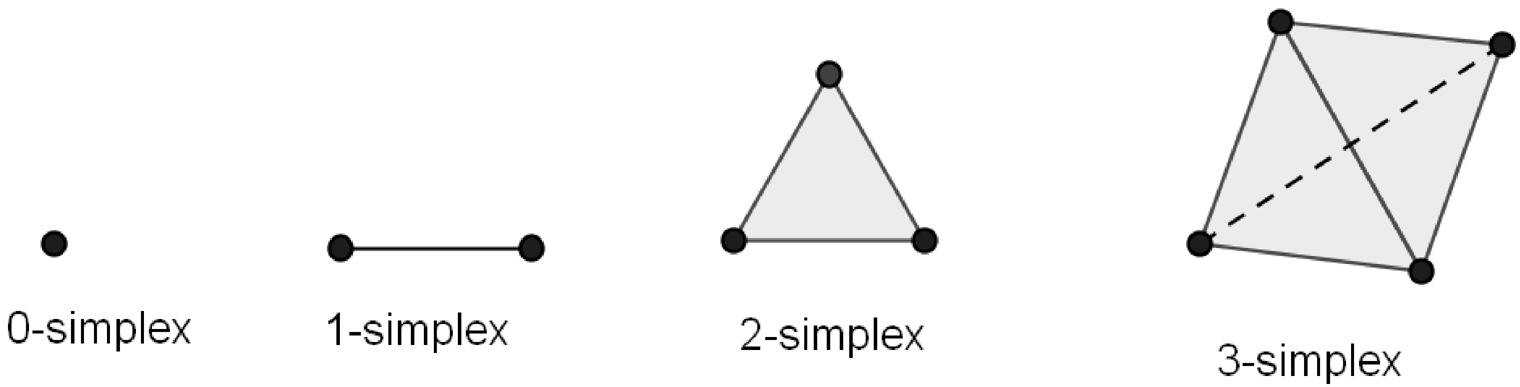

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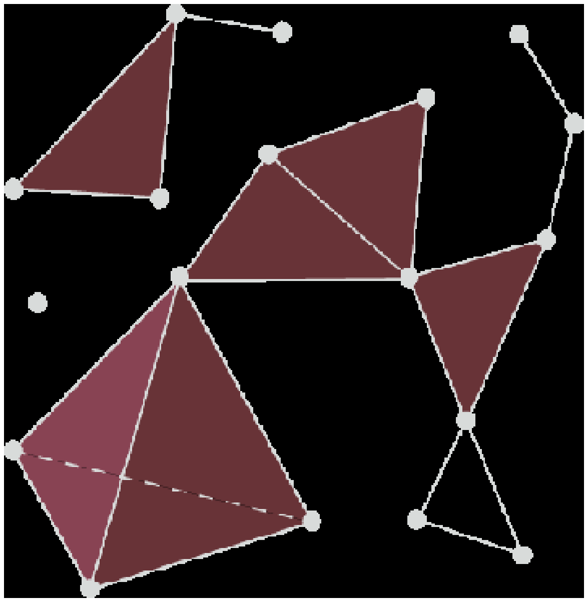

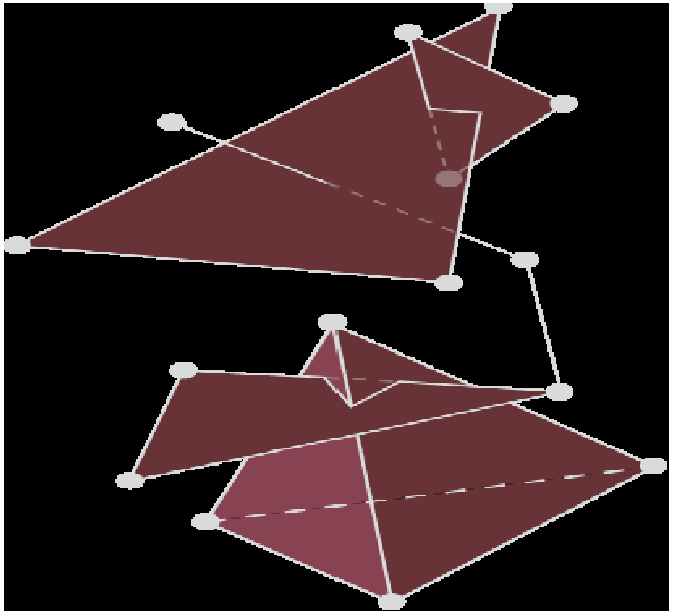

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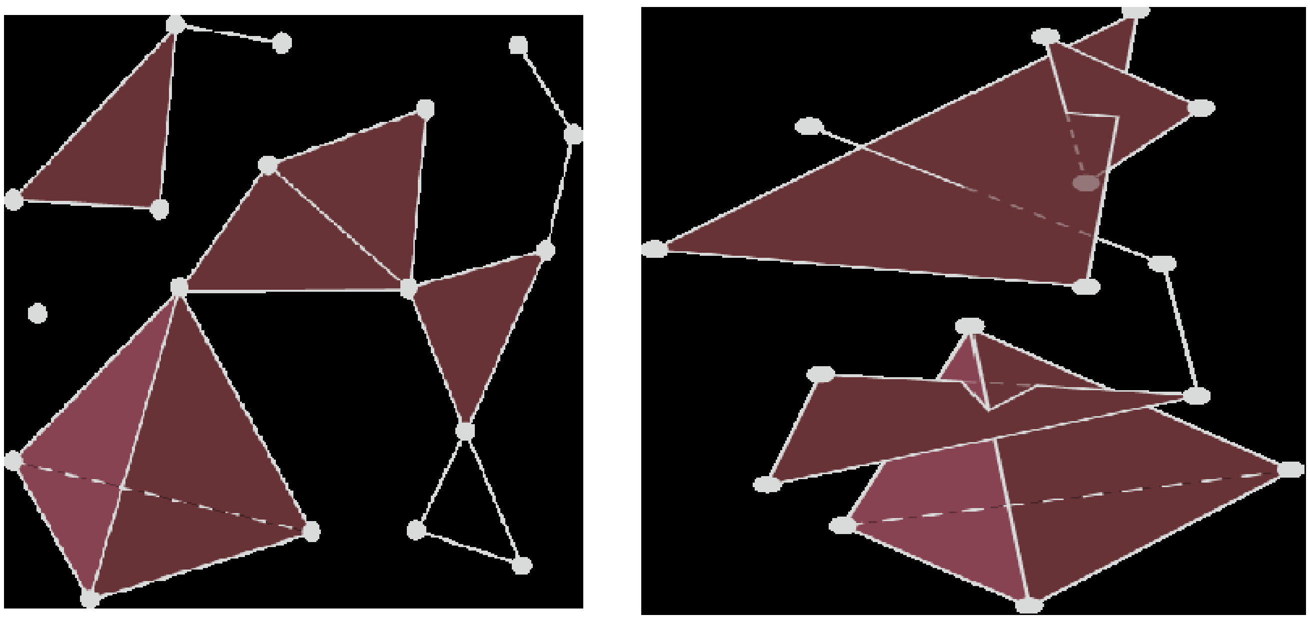

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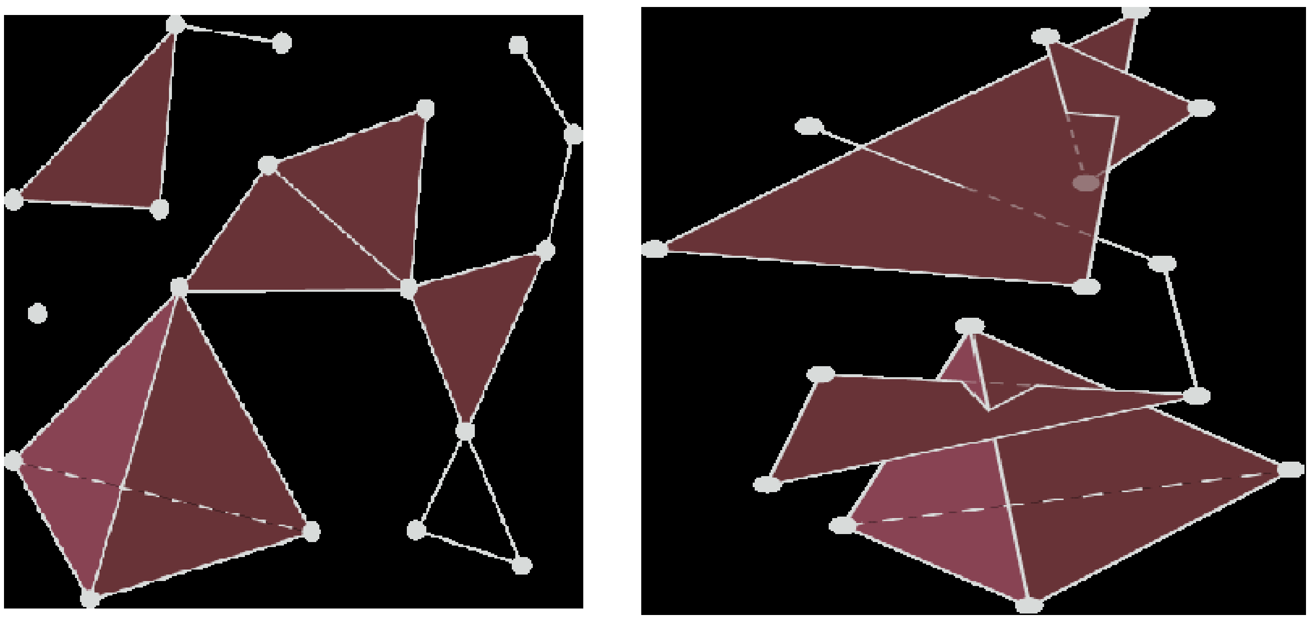

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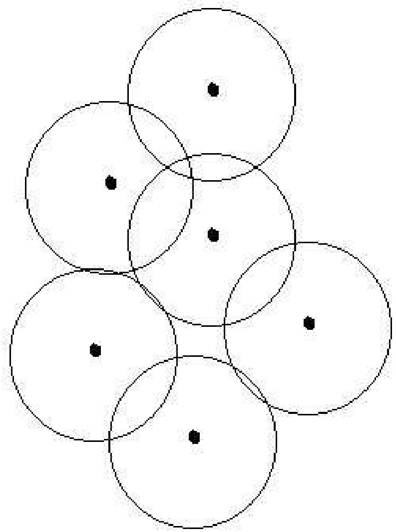

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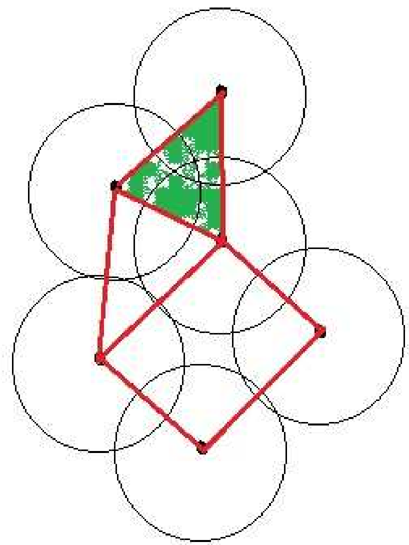

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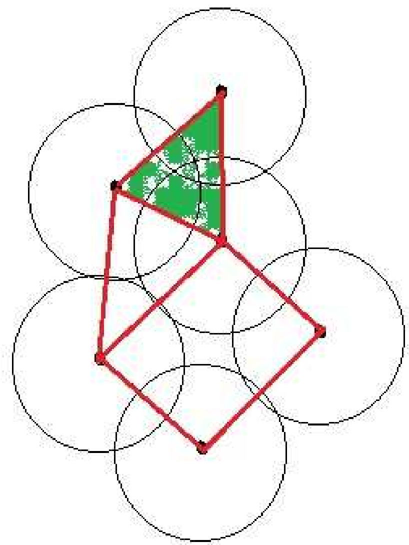

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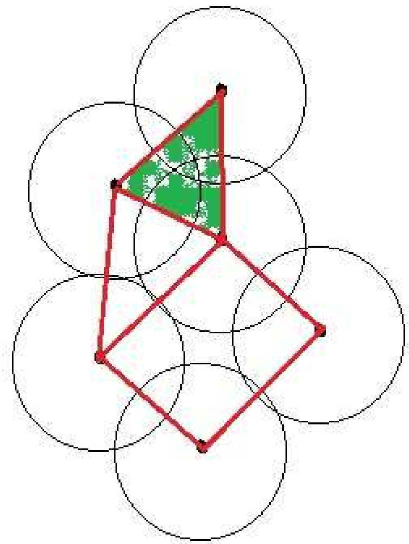

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$$\cap_{i=0}^{k} B_r(X_i) \neq \emptyset.$$

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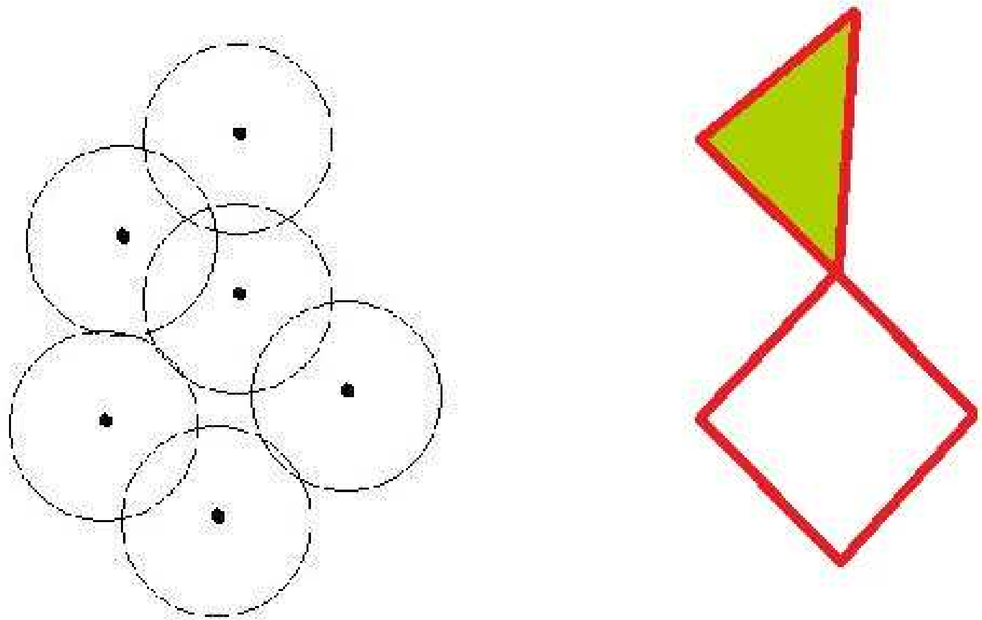

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$\Phi_n = \Phi \cap [\frac{-n^{1/d}}{2}, \frac{n^{1/d}}{2}]^d$. $S_k(\Phi_n, r)$ as $n \to \infty$?

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Clustering Condition

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$\sum_k \frac{A^k C_k}{k!} < \infty$ for any A and a little more !

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Other Examples: Poisson point processes, certain Gibbs point processes, certain Cox point processes, finite-range dependent point processes.

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Zeros of GAF and Determinantal pp : $\alpha = \frac{d-1}{d}$?

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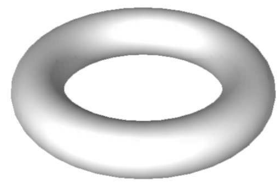

Figure 7: $\chi(T) = 0$.

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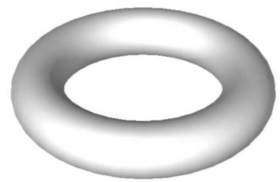

Figure 9: $\chi(T) = 0$.

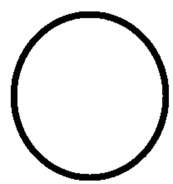

Figure 10: $\chi(S) = 0$.

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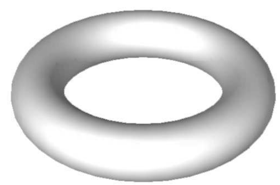

Figure 19: $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1.$

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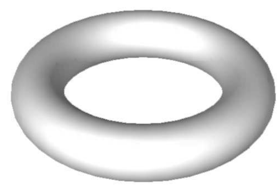

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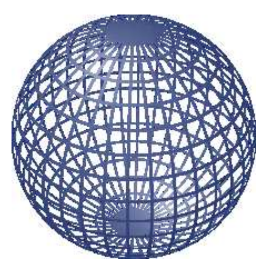

Figure 22: $\beta_0(S) = 1, \beta_1(S) = 0, \beta_2(S) = 1.$

Betti Numbers (in 1 slide)

$\beta_0(.) - \sharp$ connected components ;

$\beta_1(.) - \sharp$ two dimensional or circular holes ;

$\beta_2(.) - \sharp$ three dimensional holes or voids.

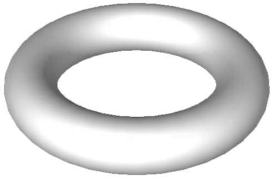

Figure 23: $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1.$

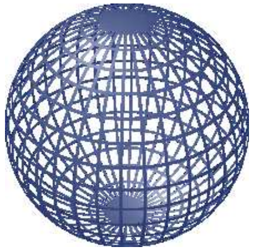

Figure 24: $\beta_0(S) = 1, \beta_1(S) = 0, \beta_2(S) = 1.$

Alexander's duality : $\beta_{d-1}(A) = \beta_0(\mathbb{R}^d \setminus A) - 1.$

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- $\rho^k(.) \equiv 1$.

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True for stationary point processes with a little different argument.

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Proof via theorem of Penrose-Yukich.

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$\beta_0(C(\mathcal{P}_n, r))$, $\quad \beta_{d-1}(C(\mathcal{P}_n, r)) = \beta_0(\mathbb{R}^d \setminus C(\mathcal{P}_n, r)) - 1$.

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