THE BURGERS SUPERPROCESS

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Abstract

We define the Burgers superprocess to be the solution of the stochastic partial differential equation

$$\frac{\partial}{\partial t} u(t,x) = \Delta u(t,x) \; - \; \lambda u(t,x) \nabla u(t,x) \; + \; \gamma \sqrt{u(t,x)}\, W(dt,dx),$$

where $t \geq 0$, $x \in \mathbb{R}$, and W is space-time white noise. Taking $\gamma = 0$ gives the classic Burgers equation, an important, non-linear, partial differential equation. Taking $\lambda = 0$ gives the super Brownian motion, an important, measure valued, stochastic process. The combination gives a new process which can be viewed as a superprocess with singular interactions. We prove the existence of a solution to this equation and its Hölder continuity, and discuss (but cannot prove) uniqueness of the solution.

Key words: Burgers equation, superprocess, stochastic partial differential equation
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1 The Burgers superprocess

The Burgers equation is a non-linear partial differential equation (PDE) that was initially introduced as a 'simplified' Navier-Stokes equation but quickly took an important place in its own right at the center of non-linear PDE theory. In one dimension, it takes the form of (1.1) below with $\gamma = 0$, and a parameter $\mu > 0$, the viscosity, is usually attached to the Laplacian Δ. The super Brownian motion (SBM) is a measure valued stochastic process, which in one dimension admits a density which is

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the solution of the *stochastic* partial differential equation (SPDE) (1.1) with $\lambda = 0$. SBM originated as a diffusion limit for a system of branching particles in [26] and has been at the center of considerable activity in Probability Theory for the last 20 years. Mixing these two terminologies, we call a solution of the following SPDE a *Burgers superprocess*:

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) \;-\; \lambda u(t,x)\nabla u(t,x) \;+\; \gamma\sqrt{u(t,x)}\; W(dt,dx). \tag{1.1}$$

Here W is space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$ [25]. The probabilistic interpretation of this equation is a model for branching particles with singular interactions, as we shall explain in a moment, once we have formulated our main result.

Theorem 1.1 *For any $\kappa > 0$ and initial condition $u(0,\cdot) \in L^1 \cap L^{2+\kappa}$ there exists at least one weak (in the probabilistic sense) solution $u(t,x)$ to the SPDE* (1.1)*. Moreover, this solution is non-negative and belongs to the space $C^{\varrho,2\varrho}(\mathbb{R}_+,\mathbb{R})$ for any $0 < \varrho < \frac{1}{4}(1 - \frac{1}{2+\kappa})$, where $C^{\alpha,\beta}(\mathbb{R}_+,\mathbb{R})$ is the space of Hölder continuous functions of order α in time and β in space. Finally, $u(t,\cdot) \in L^1(\mathbb{R})$ and $\|u(t,\cdot)\|_1$ is a Feller branching diffusion. This implies that the solution dies out in finite time.*

By way of motivation, we note that Burgers equation has entered the Probability literature in a number of ways. Among these are as the normalized limit of the asymmetric simple exclusion process [15] and via the motion of particle systems with highly local interactions. We briefly describe the latter, since this is what initially motivated us. Full details can be found in [24].

Consider a system $(X^1_t,\ldots,X^n_t)$ of n particles in $\mathbb{R}$ following the n-dimensional SDE

$$dX^i_t \;=\; dB^i_t + \frac{c}{n}\sum_{j\neq i} dL^0_t(X^i - X^j), \qquad i = 1,\ldots,n, \tag{1.2}$$

where $c > 0$, the B^i are independent Brownian motions and $L^0_t(\cdot)$ denotes local time at 0. The local time in (1.2) implies that the interaction among the individual particles is highly localized. The empirical measures of the X^i converge, as $n \to \infty$, to a deterministic limit whose density satisfies the Burgers equation with $\lambda = 2c$. The moving wave front of the solution of Burgers equation can therefore be understood as a drift due to particle interaction.

SBM, on the other hand, is a random measure arising as the limit of the empirical measure of a system of branching Brownian motions. For details, see [20]. In one dimension we can define it via its density, which is the unique (in law) solution of (1.1) with $\lambda = 0$ and branching rate $\gamma > 0$. Strong uniqueness is an open problem.

In view of the above, the Burgers superprocess, the solution of (1.1), can now be thought of as describing the infinite density limit of a system of branching Brownian motions undergoing the singular interaction (1.2). This approach also has a natural interpretation via a limit (cf. [1]) of historical super-processes with smooth interaction of the kind studied in [17]. We imagine that both of these ideas can be turned into theorems, but have not done so and so use them only as heuristics. Nevertheless, it was these heuristics that lead us to (1.1).

Returning now to Theorem 1.1, note that the Hölder continuity there is essentially the same as for the super Brownian motion, or for the Burgers SPDE with additive noise [3]. Indeed, the Burgers superprocess is closely related the nonlinear SPDE

$$\frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + \frac{\partial}{\partial x} f(t,x,u(t,x)) + \sigma(t,x,u(t,x))W(dt,dx), \tag{1.3}$$

for which existence and uniqueness on the entire real line were proven in [12] under a Lipschitz condition for σ and under the assumption $\sigma(t,x,u(t,x)) \le h(x)$ for some $h \in L^2$. The proof was based on extending the methodology developed in [5] and [11] for equations on bounded intervals. A corresponding result for the case of coloured driving noise was treated in [6]. For additive noise (σ constant) existence was proven in [3]. Our result therefore extends [12] in that our σ is both unbounded and non-Lipschitz, although it is now rather specific.

It is worth noting that the method of proof we adopt for Theorem 1.1 can also cover the other two "classical" superprocess noises, specifically those arising in the Fleming-Viot and stepping stone models with forcing noise, viz.

$$\begin{aligned} &\sqrt{u(t,x)}\,W(dt,dx) - \int u(t,x)\sqrt{u(t,y)}\,W(dt,dy)dx, && \text{(Fleming-Viot)} \\ &\gamma\sqrt{u(t,x)(1-u(t,x))}\,W(dt,dx). && \text{(stepping stone)} \end{aligned}$$

These models are treated in detail in GB's thesis [2]. However, the related parabolic Anderson problem which corresponds to the forcing noise $u(t,x)W(dt,dx)$ is not covered by our method and its existence and uniqueness remains open.

The proof of Theorem 1.1, which relies on an approach going back at least to [10], involves a spatial discretization of (1.1) to obtain a multi-dimensional stochastic differential equation (SDE). This approach has many advantages. Firstly, if we were to adopt the route of taking a Lipshitz approximation (bounded or not) to the square root in (1.1) we would gain little, since the existence of a solution even in this setting is also unknown. Secondly, our approximation is closely related to a superprocess with state space $\mathbb{Z}$, so it shares some properties with the continuum version. It should also be noted that the linear version of this model has been studied in considerable detail (e.g. [4]). Finally, this approximation provides a numerical scheme for simulation. In the deterministic setting, such a scheme is often called the *numerical method of lines*, where spatial and temporal discretization are carried out independently.

The remainder of the paper is organized as follows. In Section 2, we set up a discrete approximation to (1.1) via an infinite-dimensional SDE. In Section 3 we derive some

L^p bounds for these approximations. The tightness argument is carried out in Section 4, and the convergence of these approximations is established in Section 5. In Section 6 we discuss the issue of uniqueness (in law) of the solution of (1.1) (i.e. we explain why we have not been able to establish it) and the paper concludes with an Appendix containing some technical results needed along the way.

Finally, we want acknowledge our debts to Leonid Mytnik for a number of very helpful conversations, in particular regarding the proposed dual process, and to Roger Tribe, whose careful reading and discovery of too many minor errors in two earlier versions of this paper is greatly appreciated.

2 An infinite dimensional SDE

We start by taking $\lambda = \gamma = 1$ in (1.1), a convention that we shall adopt throughout the remainder of the paper. Since we are only interested in the existence of a solution to (1.1) scaling arguments trivially show that this simplification is unimportant.

Fix $N > 2$. Denote the rescaled integer lattice by $\mathbb{Z}_N = \{\dots, -\frac{1}{N}, 0, \frac{1}{N}, \dots\}$ and define an approximation $\{U_N\}_{N>2}$ to (1.1) via the solution, if one exists, of the infinite dimensional SDE

$$dU_N(t,x) = \mathcal{A}_N U_N(t,x)\,dt \; + \; \sqrt{U_N(t,x)_+}\, d\left(\sqrt{N} B_x(t)\right), \quad x \in \mathbb{Z}_N, \tag{2.1}$$

where the B_x are independent Brownian motions chosen to approximate the space-time white noise in (1.1). For $f : \mathbb{Z}_N \to \mathbb{R}$ the various operators in (2.1) are defined as follows:

$$\mathcal{A}_N(f) = \Delta_N f + \nabla_N F_N(f), \tag{2.2}$$

$$F_N(f)(x) = \frac{1}{3}\Big(f^2(x - \tfrac{1}{N}) + f^2(x) + f(x) f(x - \tfrac{1}{N})\Big), \tag{2.3}$$

$$\Delta_N f(x) = N^2\Big(f(x + \tfrac{1}{N}) + f(x - \tfrac{1}{N}) - 2f(x)\Big), \tag{2.4}$$

$$\nabla_N f(x) = N\Big(f(x + \tfrac{1}{N}) - f(x)\Big), \tag{2.5}$$

$$f(x)_+ = f(x)^+ = f(x) \vee 0, \qquad f(x)_- = f(x)^- = -(f(x) \wedge 0).$$

The operators Δ_N and ∇_N are, respectively, the discrete Laplacian and gradient, and the function F_N is an approximation to the square function. Define also

$$l^p_N = \Big\{ f : \mathbb{Z}_N \to \mathbb{R} : \|f\|_p^p = \frac{1}{N} \sum_{x \in \mathbb{Z}_N} |f(x)|^p < \infty \Big\}, \tag{2.6}$$

and, when $p=2$, the inner product

$$\langle f,g\rangle_N = \frac{1}{N}\sum_{x\in\mathbb{Z}_N} f(x)g(x), \qquad \text{for } f,g\in l_N^2. \tag{2.7}$$

Note that the norm in l_N^p is just the usual L^p norm if the functions f are extended to step functions on $\mathbb{R}$. The following monotonicity properties of $\mathcal{A}_N$ are crucial: for any $f\in l_N^2$

$$\begin{aligned} \langle 1,\mathcal{A}_N(f)\rangle_N &= 0, & (2.8)\\ \langle f,\mathcal{A}_N(f)\rangle_N &= -\|\nabla_N f\|_1 \le 0. & (2.9)\end{aligned}$$

Property (2.8) is a direct consequence of the fact that Δ_N and ∇_N are difference operators. Property (2.9) follows by summation by parts and from the form of F_N, which leads to a telescopic sum, the discrete analogue of $\int_{-\infty}^{\infty} u(x)(u^2(x))'dx = 0$.

Remark 2.1 *The choice of F_N above is for later convenience, for which it will be important that* (2.9) *holds. If, for example, we were to take $F_N(x)=x^2$, then* (2.9) *would hold only for non-negative f, which would cause technical problems in the proofs of Section 3.*

Theorem 2.2 *With $U_N(0,\cdot)\in l_N^1$ an initial condition for the SDE system* (2.1),

(i) There exists a unique strong solution $U_N(t,x)$ to (2.1) *which is strongly continuous in l_N^2 and such that, for any $T>0$,*

$$\begin{aligned} E\Big\{\sup_{0\le t\le T}\|U_N(t,\cdot)\|_2^2\Big\} &< \infty, & (2.10)\\ E\Big\{\|U_N(T,\cdot)\|_1\Big\} &< \infty. & (2.11)\end{aligned}$$

(ii) In addition, assume that $U_N(0,x)\ge 0$. Define

$$\begin{aligned} \Gamma_N &= \{f\in l_N^2 : \textit{for some } x\in\mathbb{Z}_N, |f(x+\tfrac{1}{N})-f(x-\tfrac{1}{N})|\ge 3N\},\\ \rho_N &= \inf\{t\ge 0 : U_N(t)\in\Gamma_N\}. \end{aligned} \tag{2.12}$$

If $U_N(0,\cdot)\notin\Gamma_N$, then, for $t\le\rho_N$, $U_N(t,x)\ge 0$ for all $x\in\mathbb{Z}_N$.

Remark 2.3 *By standard Markov process theory, if $U_N(0,\cdot)\notin\Gamma_N$, then $P\{\rho_N > 0\}=1$. We shall show in Section 4 that $\lim_{N\to\infty}P\{\rho_N\ge T\}=1$ for all $T>0$.*

PROOF. We first prove the existence of a solution via approximation by a finite system. We shall follow closely the argument of Shiga and Shimuzu [22] who consider a related model under slightly different hypotheses on the coefficients. For fixed $N>2$

and for each integer $m > 1$, set $\Lambda_N^m = \mathbb{Z}_N \cap [-m, m]$. As a first step, for given initial condition $x = \{x_i\}_{i\in\mathbb{Z}_N} \in l_N^2$, consider the following equation, defining a process $x^m(t)$ with values in l_N^2:

$$x_i^m(t) = \begin{cases} x_i \;+\; \int_0^t \mathcal{A}_N x_i^m(s)\,ds \;+\; \int_0^t \sqrt{N x_i^m(s)_+}\,dB_i(s) & i \in \Lambda_m^N, \\ x_i & i \notin \Lambda_m^N. \end{cases} \tag{2.13}$$

We retain the notation (2.3)–(2.5) for the operators that appear in (2.13), with the small change that they are now taken to have periodic boundary conditions on Λ_m^N. This will ensure that (2.8) and (2.9) still hold. In the following lemma, we establish the existence and uniqueness of a solution to (2.13), along with some of its properties. We shall then return to the proof of Theorem 2.2.

Lemma 2.4 *Let $x = \{x_i\}_{i\in\mathbb{Z}_N} \in l_N^2$. Then, for any $m > 1$, there exists a unique strong solution to the system* (2.13). *Moreover, for any $T > 0$, there exists a finite C, that depends on the initial condition x, on T and N, but not on m, such that*

$$E\Big\{\sum_{i\in\mathbb{Z}_N} |x_i^m(t)|\Big\} \le C, \qquad t \le T, \tag{2.14}$$

$$E\Big\{\sup_{0\le t\le T} \sum_{i\in\mathbb{Z}_N} x_i^m(t)^2\Big\} \le C, \tag{2.15}$$

$$\sup_{0\le t\le T} E\Big\{\Big[\sum_{i\in\mathbb{Z}_N} x_i^m(t)^2\Big]^2\Big\} \le C. \tag{2.16}$$

PROOF. Weak existence for the system (2.13) follows from the Skorokhod existence theorem, and non-explosion from the one-sided growth condition. Pathwise uniqueness can then be shown by the standard Yamada-Watanabe argument or via a local time argument as in [21]. Indeed, by a lemma of Le Gall (see Ch. IX in [21]) the local time processes at 0 for the martingales $\int_0^t \sqrt{N x_i^m(s)_+}dB_i(s)$ are 0 for each i. We use standard arguments based on Itô's formula to compute moments. Take $R > 0$ and let $\sigma_R = \inf\{t : |x^m(t)| = R\}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^{2m+1}$. Define $x^R(t) = x^m(t \wedge \sigma_R)$. By Tanaka's formula and the form of the coefficients of (2.13),

$$E\Big\{\sum_{i\in\mathbb{Z}_N} x_i^R(t)_+\Big\} \le \sum_{i\in\mathbb{Z}_N} (x_i)_+ + c_1 \int_0^t \sum_{i\in\mathbb{Z}_N} E\Big\{x_i^R(s)_+\Big\}ds + c_2 \int_0^t \sum_{i\in\mathbb{Z}_N} E\Big\{x_i^R(s)^2\Big\}ds.$$

Furthermore, by (2.9) and Itô's formula,

$$E\Big\{\sum_{i\in\mathbb{Z}_N} x_i^R(t)^2\Big\} \;\le\; \sum_{i\in\mathbb{Z}_N} x_i^2 + N\int_0^t \sum_{i\in\mathbb{Z}_N} E\left\{x_i^R(s)_+\right\}ds. \tag{2.17}$$

Summing these two inequalities, an application of Gronwall's inequality to $E\Big\{\sum_{i\in\mathbb{Z}_N}(x_i^R(s)_+ + x_i^R(s)^2)\Big\}$ gives that, for any T, there exist constants c_3 and c_4, depending only on $\sum_{i\in\mathbb{Z}_N} x_i$, $\sum_{i\in\mathbb{Z}_N} x_i^2$, N and T, such that

$$E\Big\{\sum_{i\in\mathbb{Z}_N} x_i^R(t)_+\Big\} \le c_3, \qquad E\Big\{\sum_{i\in\mathbb{Z}_N} x_i^R(t)^2\Big\} \le c_4 \qquad \text{for } t \le T. \tag{2.18}$$

Since this bound does not depend on R, we can let $R \to \infty$ so that $P\{\sigma_R \le T\} \to 0$, which then yields the same bounds for $x^m(t)$. Using these and a further application of Tanaka's formula for $x_i^m(t)_-$ gives (2.14). The inequality (2.15) now follows from (2.18) and an application of Burkholder's inequality. The final bound (2.16) follows from (2.14) and (2.15) via reasonably straightforward calculations. □

Returning to the proof of Theorem 2.2, and following the argument of [22], we shall establish the following bounds, for $T > 0$, $t, s < T$, $|t-s| \le 1$ and constants $C_1,\ C_2$:

$$\sup_m E\left\{\sup_{0\le t\le T} \|x^m(t)\|_2^2\right\} \le C_1, \tag{2.19}$$

$$\sup_m E\left\{\|x^m(t) - x^m(s)\|_2^2\right\} \le C_2|t-s|. \tag{2.20}$$

These bounds correspond exactly to (2.7)–(2.8) of [22], and once proved, their argument establishes the existence of a unique strong solution to (2.1). We rescale the norms in Lemma 2.4 by the constant $\frac{1}{N}$. Note that $U_N(0,\cdot) \in l_N^1$ implies that $U_N(0,\cdot) \in l_N^2$. Then (2.19) follows from (2.15), since

$$E\left\{\sup_{0\le t\le T} \|x^m(t)\|_2^2\right\} \le E\left\{\frac{1}{N}\sum_{i\notin\Lambda_m^N} x_i^2\right\} \;+\; E\left\{\sup_{0\le t\le T}\frac{1}{N}\sum_{i\in\Lambda_m^N} (x_i^m(t))^2\right\},$$

which is bounded independently of m by Lemma 2.4. As for (2.20), note that

$$\begin{aligned}
\|x^m(t) - x^m(s)\|_2^2 \le & \frac{2}{N}\sum_{i\in\Lambda_m^N}\left(\int_s^t \Delta_N x_i^m(u) + \nabla_N F_N(x_i^m(u))du\right)^2 \\
& + \frac{2}{N}\sum_{i\in\Lambda_m^N}\left(\int_s^t \sqrt{N x_i^m(u)_+}\, dB_i(u)\right)^2 \\
\le & \ \frac{2}{N}\left\{\int_s^t \left[\sum_{i\in\Lambda_m^N} (\Delta_N x_i^m(u) + \nabla_N F_N(x_i^m(u)))^2\right]^{1/2} du\right\}^2 \\
& + \frac{2}{N}\sum_{i\in\Lambda_m^N}\left(\int_s^t \sqrt{N x_i^m(u)_+}\, dB_i(u)\right)^2,
\end{aligned} \tag{2.21}$$

where the last inequality follows from Minkowski's inequality in the form ([16] p.41)

$$\left(\int_Y \left(\int_X f(x,y)\nu(dx)\right)^p \mu(dy)\right)^{1/p} \leq \int_X \left(\int_Y f(x,y)^p \mu(dy)\right)^{1/p} \nu(dx).$$

Taking expectations, the bound $\sum_{i\in\Lambda_m^N} (\Delta_N x_i + \nabla_N F_N(x)_i)^2 \leq 36N^4|x|_2^2 + 4N^2|x|_2^4$, Minkowski's inequality and (2.16) control the first term in (2.21), and the second can be bounded by (2.14).To complete the proof of (i), it is enough to note that the bounds (2.10) and (2.11) follow, respectively, from the bounds (2.19) and (2.14) and Fatou's inequality.

Turning now to part (ii) of the theorem, to establish the non-negativity of $U_N(t,x)$, we first rewrite the drift term as

$$\begin{aligned}\mathcal{A}_N f(x) = &\frac{N}{3}\Big[f(x+\tfrac{1}{N}) + f(x-\tfrac{1}{N})\Big]\Big[3N + f(x+\tfrac{1}{N}) - f(x-\tfrac{1}{N})\Big] \\ &+\frac{N}{3} f(x)\Big[f(x+\tfrac{1}{N}) - f(x-\tfrac{1}{N}) - 6N\Big].\end{aligned}$$

We use a pathwise argument and take $t \leq \rho_N$ to obtain, by Tanaka's formula,

$$\begin{aligned}\sum_{x\in\mathbb{Z}_N} U_N(t,x)^- = &-\frac{N}{3}\int_0^t \sum_{x\in\mathbb{Z}_N} \mathbb{1}_{\{U_N(s,x)<0\}}\Big[U_N(s,x+\tfrac{1}{N})^+ + U_N(s,x-\tfrac{1}{N})^+\Big] \\ &\qquad\times\Big[3N + U_N(s,x+\tfrac{1}{N}) - U_N(s,x-\tfrac{1}{N})\Big]ds \\ &+\frac{N}{3}\int_0^t \sum_{x\in\mathbb{Z}_N} \mathbb{1}_{\{U_N(s,x)<0\}}\Big[U_N(s,x+\tfrac{1}{N})^- + U_N(s,x-\tfrac{1}{N})^-\Big] \\ &\qquad\times\Big[3N + U_N(s,x+\tfrac{1}{N}) - U_N(s,x-\tfrac{1}{N})\Big]ds \\ -\frac{N}{3}\int_0^t \sum_{x\in\mathbb{Z}_N} &\mathbb{1}_{\{U_N(s,x)<0\}} U_N(s,x)\Big[U_N(s,x+\tfrac{1}{N}) - U_N(s,x-\tfrac{1}{N}) - 6N\Big]ds \\ &\triangleq A_1 + A_2 + A_3.\end{aligned}$$

If $U_N(s) \notin \Gamma_N,\ 0 \leq s \leq t$ then it is easy to see that A_1 and A_3 are negative and

$$A_2 \leq \frac{N}{3}\int_0^t \sum_{x\in\mathbb{Z}_N} \mathbb{1}_{\{U_N(s,x)<0\}}\Big[U_N(s,x+\tfrac{1}{N})^- + U_N(s,x-\tfrac{1}{N})^-\Big] \times 6N.$$

An application of Gronwall's inequality then shows that $\sum_{x\in\mathbb{Z}_N} U_N(t,x)^- = 0$, for $t \leq \rho_N$, which shows that the process stays non-negative, at least up to time ρ_N. □

3 L^p bounds for the semi-discrete superprocess

In this section, we derive some preliminary, uniform (in N), L^p bounds for our approximating processes. The reason for our restrictions regarding the model (i.e. the square root for the noise term and the square for the nonlinearity) should become clear after reading the following proofs. As is standard in the SPDE literature, our bounds will depend on Green's functions.

Let $p_N(t,x,y),\ x,y\in\mathbb{Z}_N$ be the semigroup generated by the discrete Laplacian Δ_N, so that p_N solves

$$\frac{\partial}{\partial t}p_N(t,x,y)=\Delta_N p_N(t,x,y),\quad p_N(0,x,y)=\delta_{xy},\quad x,y\in\mathbb{Z}_N. \tag{3.1}$$

We define the Green's function and its discrete derivative as follows, for $x,y\in\mathbb{Z}_N$.

$$G_N(t,x,y)=Np_N(t,x,y), \tag{3.2}$$
$$G_N'(t,x,y)=\nabla_N^- G_N(t,x,y)\triangleq N(G_N(t,x,y)-G_N(t,x,y-\tfrac{1}{N})). \tag{3.3}$$

Note that G_N is an approximation to the Gaussian kernel. The following lemma, a proof of which is given in the Appendix, summarizes the L^p properties of the Green's function, which are essentially the same as their Gaussian counterparts.

Lemma 3.1 *With G_N and G_N' as defined above, and $\|\cdot\|_p$ defined in (2.6), the following bounds hold uniformly in N for $s<t\le T$ and for all $x,y\in\mathbb{Z}_N$. The constant C varies from line to line and may depend on p, ρ, α and T.*

$$\|G_N(t,x,\cdot)\|_p\le Ct^{-\frac{1}{2}(1-\frac{1}{p})}\qquad \text{for } 1\le p\le 2, \tag{3.4}$$
$$\|G_N'(t,\cdot,y)\|_p\le Ct^{-1+\frac{1}{2p}}\qquad \text{for } p\ge 1. \tag{3.5}$$

If $1\le p\le 2$ and $0<\varrho<1$, then

$$\|G_N(t,\cdot,x)-G_N(s,\cdot,y)\|_p\le C\Big(|t-s|^{\varrho}+|y-x|^{2\varrho}\Big)s^{-\frac{1}{2}-\varrho+\frac{1}{2p}}. \tag{3.6}$$

If $1\le p\le 2$ and $1>\alpha>\varrho+\frac{1}{2}(1-\frac{1}{p})>0$, then

$$\begin{aligned}&\int_0^t \|(t-u)^{\alpha-1}G_N(t-u,\cdot,x)-(s-u)^{\alpha-1}G_N(s-u,\cdot,y)\mathbb{1}_{(u\le s)}\|_p\,du\\&\quad\le C\Big(|t-s|^{\varrho}+|y-x|^{2\varrho}\Big).\end{aligned} \tag{3.7}$$

Our next lemma establishes some important L^p properties for a certain stochastic convolution.

Lemma 3.2 *Let $\{B_y(t)\}_{y\in\mathbb{Z}_N}$ be a collection of independent Brownian motions on a probability space $\Big(\Omega,\mathcal{F},\mathcal{F}_t,P\Big)$, and let u_N be an l^2_N valued, non-negative, continuous and $\mathcal{F}_t$ adapted stochastic process for which*

$$E\Big\{\sup_{0\le t\le T}\|u_N(t)\|_1^q\Big\}\le K \tag{3.8}$$

for some $q>2$ (and therefore for all $q'\le q$) and $K>0$. Define

$$\eta_N(t,x)=\frac{1}{\sqrt{N}}\sum_{y\in\mathbb{Z}_N}\int_0^t G_N(t-s,x,y)\sqrt{u_N(s,y)}dB_y(s). \tag{3.9}$$

Then, for any $T>0$, there exist constants K_1,K_2,K_3, independent of N, such that

$$E\Big\{\sup_{t\le T}\|\eta_N(t)\|_2^p\Big\}\le K_1 \qquad \text{for } p\le 2q, \tag{3.10}$$

$$E\Big\{\sup_{t\le T}\|\eta_N(t)\|_p\Big\}\le K_2 \qquad \text{for } p<q, \tag{3.11}$$

$$E\Big\{\|\eta_N(t)\|_p^p\Big\}\le K_3 \qquad \text{for } t\le T,\ p<2q. \tag{3.12}$$

PROOF. To prove (3.10) we use the so-called "factorization formula" (e.g. [7] p128) which uses the semigroup property of G_N and implies that η_N can be written as

$$\eta_N(t,x)=\frac{\sin(\pi\alpha)}{\pi}\int_0^t(t-s)^{\alpha-1}\frac{1}{N}\sum_{z\in\mathbb{Z}_N}G_N(t-s,x,z)Y_N(s,z)ds, \tag{3.13}$$

where

$$Y_N(s,z)=\int_0^s\frac{1}{\sqrt{N}}\sum_{y\in\mathbb{Z}_N}(s-v)^{-\alpha}G_N(s-v,z,y)\sqrt{u_N(v,y)}dB_y(v). \tag{3.14}$$

Note that by Burkholder's inequality for L^2 valued martingales ([18] p213), and using successively Young's inequality, (3.4), Minkowski's inequality, and finally (3.8), we have, for $\alpha<1/4$ and $p/2\le q$, and uniformly in $s\le T$,

$$\begin{aligned}
E\left\{\|Y_N(s)\|_2^p\right\}&\le CE\Big\{\Big[\int_0^s\frac{1}{N^2}\sum_{x,y\in\mathbb{Z}_N}(s-u)^{-2\alpha}G_N(s-u,x,y)^2u_N(u,y)du\Big]^{\frac{p}{2}}\Big\}\\
&\le CE\Big\{\Big[\int_0^s(s-u)^{-2\alpha-1/2}\|u_N(s,\cdot)\|_1du\Big]^{\frac{p}{2}}\Big\}\\
&\le C\Big[\int_0^s(s-u)^{-2\alpha-1/2}\Big[E\Big\{\|u_N(s,\cdot)\|_1^{\frac{p}{2}}\Big\}\Big]^{\frac{2}{p}}du\Big]^{\frac{p}{2}}\\
&<C.
\end{aligned} \tag{3.15}$$

Returning to (3.10), note that by the Young and Hölder inequalities, for $\alpha > 1/p$,

$$\begin{aligned} E\Big\{\sup_{t\le T}\|\eta_N(t)\|_2^p\Big\} &\le E\Big\{\Big[\sup_{t\le T}\int_0^t (t-s)^{\alpha-1}\|Y_N(s)\|_2 ds\Big]^p\Big\} \\ &\le CE\Big\{\sup_{t\le T}\Big[\int_0^t (t-s)^{(\alpha-1)\frac{p}{p-1}}ds\Big]^{p-1}\int_0^t \|Y_N(s)\|_2^p ds\Big\} \\ &\le CE\int_0^T \|Y_N(s)\|_2^p ds, \end{aligned}$$

from which (3.10) follows by (3.15) provided that $p > 4$ with $p/2 \le q$. To prove (3.11) take $\gamma > 0$, $\beta = \frac{2p}{p+2}$ and note that the Young, Hölder and Jensen inequalities imply

$$\begin{aligned} E\Big\{\sup_{t\le T}\|\eta_N(t)\|_p\Big\} &\le CE\Big\{\sup_{t\le T}\int_0^t (t-s)^{\alpha-1}\|G_N(t-s)\|_\beta\|Y_N(s)\|_2 ds\Big\} \\ &\le CE\Big\{\sup_{t\le T}\int_0^t (t-s)^{\alpha-\frac{3}{2}+\frac{1}{2\beta}}\|Y_N(s)\|_2 ds\Big\} \\ &\le C\Big[\int_0^T s^{(\alpha-\frac{3}{2}+\frac{1}{2\beta})\gamma}ds\Big]^{\frac{1}{\gamma}} E\Big\{\Big[\int_0^T \|Y_N(s)\|_2^{\frac{\gamma}{\gamma-1}}ds\Big]^{\frac{\gamma-1}{\gamma}}\Big\} \\ &\le C\Big[\int_0^T E\|Y_N(s)\|_2^{\frac{\gamma}{\gamma-1}}ds\Big]^{\frac{\gamma-1}{\gamma}}, \end{aligned} \tag{3.16}$$

which will prove (3.11), provided that $\alpha < 1/4$ and $\frac{\gamma}{2(\gamma-1)} \le q$ in order to use (3.15), and $\alpha > \frac{3}{2} - \frac{1}{2\beta} - \frac{1}{\gamma}$ in the above calculation. This leads to $2p < \frac{\gamma}{\gamma-1}$.
The proof of (3.12) is similar up to the second line of (3.16) and concludes with a further application of Minkowski's inequality. This gives

$$E\left\{\|\eta_N(t,x)\|_p^p\Big\} \le C\Big[\int_0^t (t-s)^{\alpha-\frac{3}{2}+\frac{1}{2\beta}}\Big[E\Big\{\|Y_N(s)\|_2^p\Big\}\Big]^{1/p}ds\Big]^p.$$

We now demand that $\alpha > 1/2 - (p+2)/4p$, which holds for any $p > 0$, and $\alpha < 1/4,\ p < 2q$, which we required for (3.15). □

Our next preparatory step involves finding a L^2 bound for the non-negative approximating process. Thus, let $f_N \in l_N^2$ be a sequence of non-negative functions for which

$$\begin{cases} \|f_N\|_1 + \|f_N\|_{2+\kappa} \quad \text{is bounded, uniformly in } N, \text{for some } \kappa > 0, \\ \bar{f}_N \to f \quad \text{in } C(\mathbb{R}), \quad \text{where } \bar{f}_N \text{ is the polygonal extension of } f_N. \end{cases} \tag{3.17}$$

Recall the definition of Γ_N given in (2.12), and let $\widetilde{U}_N(t,x)$ be the solution to

$$
\begin{aligned}
d\widetilde{U}_N(t,x) = & \Big[\Delta_N \widetilde{U}_N(t,x) + \mathbb{1}_{[0,\rho_N]}(t)\nabla_N F_N(\widetilde{U}_N)(t,x)\Big]\, dt \\
& \qquad + \sqrt{\widetilde{U}_N(t,x)}\, d\Big(\sqrt{N} B_x(t)\Big), \quad x \in \mathbb{Z}_N, \qquad (3.18)\\
\rho_N = & \inf\{t : \widetilde{U}_N(t) \in \Gamma_N\}. \qquad (3.19)
\end{aligned}
$$

Lemma 3.3 *Let f_N satisfy* (3.17)*. Then there exists a unique solution $\widetilde{U}_N(t) \in l^2_N$ to* (3.18)*, with initial value $\widetilde{U}_N(0) = f_N$ and which is non-negative. Moreover, for any $T > 0$ and $p > 0$, there exists K, independent of N, such that, for this solution,*

$$
E\Big\{\sup_{0\le t\le T} \|\widetilde{U}_N(t)\|_1^p\Big\} \le K. \tag{3.20}
$$

Finally, the following representation holds, with G_N and G'_N defined by (3.2)–(3.3)*.*

$$
\begin{aligned}
\widetilde{U}_N(t,x) = & \frac{1}{N}\sum_{y\in\mathbb{Z}_N} G_N(t,x,y)f_N(y) - \frac{1}{N}\sum_{y\in\mathbb{Z}_N}\int_0^{t\wedge\rho_N} G'_N(t-s,x,y)F_N\Big(\widetilde{U}_N(s)\Big)(y)ds \\
& + \frac{1}{\sqrt{N}}\int_0^t \sum_{y\in\mathbb{Z}_N} G_N(t-s,x,y)\sqrt{\widetilde{U}_N(y,s)}dB_y(s) \\
\triangleq\ & I_N(t,x) + D_N(t,x) + \eta_N(t,x). \qquad (3.21)
\end{aligned}
$$

PROOF. Note that $\widetilde{U}_N$ is well defined, as it solves equation (2.1) for $t \le \rho_N$, and afterwards the same equation but without the term in $\nabla_N F_N$ for $t \ge \rho_N$. (i.e. It becomes a 'super random walk'.) In addition, by Theorem 2.2, $\widetilde{U}_N(t) \ge 0$ and $\widetilde{U}_N(t) \in l^1_N$. Therefore, if we introduce the total mass process $M_N(t) \triangleq \langle 1, \widetilde{U}_N(t)\rangle_N = \|\widetilde{U}_N(t)\|_1$ then, by (2.8), $M_N(t)$ is a martingale with $d\langle M_N\rangle_t = M_N(t)dt$. Thus, as for the super random walk and the super Brownian motion, $M_N(t)$ is a Feller branching diffusion, and it is well known that $E\Big\{\sup_{0\le t\le T} M_N(t)^p\Big\} < K$, for some $K > 0$ that depends on the moments of the initial condition, T and p but not N. This and condition (3.17) establish (3.20). The proof of (3.21) follows as for the case without ρ_N. □

Our next step is to establish the bounds for $\|\widetilde{U}_N\|_p$ that will be the main ingredients in the proof of tightness. We have just established their validity for $p = 1$, uniformly in N, and, by Theorem 2.2, for $p = 2$ for fixed N. The main difficulty will lie in proving the uniformity of the latter in N. We proceed as in [12] and have now come to the point where our method requires the quadratic non-linearity assumption in (1.1).

Let $\eta_N(t,x)$, $t \ge 0$, $x \in \mathbb{R}$, be the random field defined in Lemma 3.2, taking $u_N = \widetilde{U}_N$, and set $v_N = \widetilde{U}_N - \eta_N$. Then it is easy to see that v_N satisfies the equation

$$
\frac{\partial}{\partial t} v_N(t,x) = \Delta_N v_N(t,x) + \mathbb{1}_{[0,\rho_N]}(t)\nabla_N F_N(v_N(t) + \eta_N(t))(x), \qquad x \in \mathbb{Z}_N. \tag{3.22}
$$

Proposition 3.4 *Let $v_N = \widetilde{U}_N - \eta_N$ be as above. Then*

$$\|v_N(t)\|_2^2 \leq \left[\|v_N(0)\|_2^2 + \frac{1}{4}\int_0^t \|\eta_N(s)\|_4^4\right] \exp\left[\frac{1}{2}\int_0^t \|\eta_N(s)\|_2^4 ds\right].$$

PROOF. ¿From (3.22), the chain rule and summation by parts lead to

$$\begin{aligned}\|v_N(t)\|_2^2 &= \|f_N\|_2^2 - 2\int_0^t \|\nabla^- v_N(s)\|_2^2 ds - \int_0^{t\wedge\rho_N} \langle \nabla_N^- v_N(s), F_N(v_N(s)+\eta_N(s))\rangle_N ds \\ &\triangleq \|v_N(0)\|_2^2 - 2\int_0^t \|\nabla^- v_N(s)\|_2^2 ds - I_1(t) - I_2(t) - I_3(t),\end{aligned}$$

where the three integrals correspond to the the three terms in the decomposition

$$\begin{aligned}F_N(f+g)(x) &= F_N(f)(x) + F_N(g)(x) \\ &\quad + \frac{1}{3}\Big[f(x)\Big(2g(x) + g(x-\tfrac{1}{N})\Big) + f(x-\tfrac{1}{N})\Big(g(x) + 2g(x-\tfrac{1}{N})\Big)\Big].\end{aligned}$$

¿From the definition of F_N we have $I_1 \triangleq \int_0^{t\wedge\rho_N} \langle \nabla_N^- v_N(s), F_N(v_N(s))\rangle_N ds = 0$. Also,

$$\begin{aligned}I_2(t) &\triangleq \int_0^{t\wedge\rho_N} \langle \nabla_N^- v_N(s), F_N(\eta_N(s))\rangle_N ds \\ &\leq \frac{1}{3}\int_0^{t\wedge\rho_N} \|\nabla^- v_N(s)\|_2 \Big[2\|\eta_N(s)\|_4^2 + \Big(\langle \eta_N^2(s,\cdot), \eta_N^2(s,\cdot+\tfrac{1}{N}\rangle_N\Big)^{1/2}\Big] \\ &\leq \int_0^{t\wedge\rho_N} \|\nabla^- v_N(s)\|_2 \|\eta_N(s)\|_4^2 ds \\ &\leq \int_0^t \|\nabla^- v_N(s)\|_2^2 ds + \frac{1}{4}\int_0^t \|\eta_N(s)\|_4^4 ds,\end{aligned}$$

where we used the Cauchy-Schwartz inequality in the third line. To bound I_3 we apply Lemma 7.1 to each term in the next line to find

$$\begin{aligned}I_3(t) &\triangleq \int_0^{t\wedge\rho_N} \frac{1}{3N}\sum_{x\in\mathbb{Z}_N} \nabla_N^- v_N(s,x)\Big\{v_N(s,x)\Big(2\eta_N(s,x) + \eta_N(s,x+\tfrac{1}{N})\Big) \\ &\qquad + v_N(s,x+\tfrac{1}{N})\Big(\eta_N(s,x) + 2\eta_N(s,x+\tfrac{1}{N})\Big)\Big\} ds \\ &\leq \int_0^t \|\nabla_N v_N(s)^-\|_2^2 ds + 2\int_0^t \|\eta_N(s)\|_2^4 \|v_N(s)\|_2^2 ds.\end{aligned}$$

Putting everything together, we have

$$\|v_N(t)\|_2^2 \leq \|v_N(0)\|_2^2 + \frac{1}{4}\int_0^t \|\eta_N(s)\|_4^4 ds + 2\int_0^t \|\eta_N(s)\|_2^4 \|v_N(s)\|_2^2 ds.$$

The result then follows from Bellman's inequality. □

¿From this proposition, Lemma 3.2 and Lemma 3.3, we have the following L^2 estimate on the solution to the discrete Burgers SPDE.

Corollary 3.5 *Let $\widetilde{U}_N$ be the process defined by* (3.18)*, with initial condition f_N satisfying* (3.17)*. Then, uniformly in N,*

$$E\left\{\log_+\left(\sup_{t\leq T}\|\widetilde{U}_N(t)\|_2\right)\right\}\leq K. \tag{3.23}$$

We finish this section by deriving a L^q estimate for $\widetilde{U}_N$ for $q>2$.

Proposition 3.6 *Let $\widetilde{U}_N(t)$ be a solution of* (3.18) *with $f_N=\widetilde{U}_N(0)$ satisfying* (3.17) *for some $\kappa>0$. Then, for any $T>0$ and $q\leq 2+\kappa$ there is a constant K such that*

$$\sup_N E\Big\{\log_+\Big(\sup_{t\leq T}\|\widetilde{U}_N(t)\|_q\Big)\Big\}<K. \tag{3.24}$$

PROOF. In order to establish (3.24) we shall show that the bound is satisfied for each of the three terms corresponding to I_N, D_N and η_N in (3.21). By the condition on the initial value, the result for the I_N term is an immediate consequence of Young's inequality. For the D_N term, note that by Young's inequality and (3.5) we have

$$\begin{aligned}\|D_N(t)\|_q &\leq \int_0^{t\wedge\rho_N}\|G'_N(t-s,x,\cdot)\|_q\|F_N(\widetilde{U}_N(s))\|_1 ds\\ &\leq C\int_0^{t\wedge\rho_N}(t-s)^{-1+\frac{1}{2q}}\|\widetilde{U}_N(s)\|_2^2 ds\\ &\leq C\int_0^t (t-s)^{-1+\frac{1}{2q}}\|\widetilde{U}_N(s)\|_2^2 ds.\end{aligned}$$

Therefore, there exists a constant C, independent of N, such that

$$\log_+\Big(\sup_{t\leq T}\|D_N(t)\|_p\Big)\leq C\log_+ T+2C\log_+\Big(\sup_{t\leq T}\|\widetilde{U}_N(t)\|_2\Big),$$

and so the term corresponding to D_N can now be bounded via (3.23). The bound for η_N follows from (3.11) and Jensen's inequality. □

4 Tightness of the approximating processes

In this section we shall show that the sequence of processes $\widetilde{U}_N$, or rather its spatial polygonal interpolation, is tight in $C = C([0,\infty), C(\mathbb{R}))$. We identify this space with $C([0,\infty)\times\mathbb{R})$ endowed with the topology of uniform convergence on compacts.

For a $v = v(t,x) \in C$ and $K > 0,\ \delta > 0$, define its modulus of continuity as

$$w_\delta^K(v) \triangleq \sup\{|v(t,x) - v(s,y)| : \ |x-y| \vee |t-s| \le \delta \text{ and } \ |x|, |y|, t, s \le K\}. \quad (4.1)$$

The next lemma gives standard criteria for tightness (e.g. Chapter XIII of [21]).

Lemma 4.1 *A sequence $\{P_n\}$ of probability measures on C is tight if, and only if, the following two conditions are satisfied:*

C1: For every $\epsilon, M > 0$ there exist $A > 0$ and $n_0 \ge 0$ such that

$$P_n\Big\{ \sup_{|x|\le M} |v(0,x)| > A\Big\} \le \epsilon, \ \textit{for every } n \ge n_0.$$

C2: For every $\zeta, \epsilon, K > 0$ there exists a $\delta > 0$ and $n_0 \ge 0$ such that

$$P_n\Big\{ w_\delta^K(v) > \zeta\Big\} \le \epsilon, \ \textit{for every } n \ge n_0.$$

We can now state the main result of this section.

Theorem 4.2 *Let $\widetilde{U}_N(t)$ be as defined in* (3.18) *with $\widetilde{U}_N(0,x) = f_N(x)$ which satisfies* (3.17)*, and let $\bar{U}_N(t,x)$ be its continuous extension to $x \in \mathbb{R}$ as defined by polygonal interpolation. Then the sequence of processes $\{\bar{U}_N\}_{N\in\mathbb{N}}$ is tight in $C([0,\infty)\times\mathbb{R})$.*

PROOF. Note first that, for each N, it suffices to limit the supremum in (4.1) to $x, y \in \mathbb{Z}_N$, as the resulting restricted modulus of continuity bounds, up to a constant factor, the unrestricted one. We shall see that in all our bounds the spatial increments need not be restricted to $|x|, |y| < K$, so we shall require only that $s, t < T$. We still denote the modulus of continuity with these two minor changes by ω^T. Now consider the tightness of each of (the extension of) the terms in (3.21) separately. The term I_N involves only $\widetilde{U}_N(0,x)$ and by a standard result (e.g. [10])it converges in C.

To show the tightness of η_N we use Lemma 4.1. The first condition is trivial since $\eta(0) = 0$. Fix $T, \epsilon, \zeta > 0$, and $\kappa > 0$ such that (3.17) is satisfied. For $R > \|f\|_{2+\kappa}$, define a sequence of stopping times $\tau_N = \inf\{t : \|\widetilde{U}_N(t)\|_{2+\kappa} \ge R\}$. By Proposition 3.6 and Markov's inequality, we can choose R such that

$$\sup_N P\{\tau_N \leq T\} < \epsilon/2. \tag{4.2}$$

Let

$$\widetilde{\eta}_N(t,x) \triangleq \frac{1}{\sqrt{N}} \sum_{z\in\mathbb{Z}_N} \int_0^t G_N(t-u,x,z) \mathbb{1}_{\{\tau_N\geq u\}} \sqrt{\widetilde{U}_N(u,z)} dB_z(u).$$

Since it is trivial that $\eta_N(t\wedge\tau_N, x) = \widetilde{\eta}_N(t,x)$, we can write

$$\begin{aligned} P\Big\{w_\delta^T(\eta_N) > \zeta\Big\} &= P\Big\{\{w_\delta^T(\eta_N) > \zeta\} \cap \{\tau_N > T\}\Big\} + P\Big\{\{w_\delta^T(\eta_N) > \zeta\} \cap \{\tau_N \leq T\}\Big\} \\ &\leq P\Big\{w_\delta^T(\widetilde{\eta}_N) > \zeta\Big\} + \epsilon/2. \end{aligned}$$

Therefore, we have to show that we can find δ such that

$$\sup_N P\Big\{w_\delta^T(\widetilde{\eta}_N) > \zeta\Big\} \leq \epsilon/2. \tag{4.3}$$

To prove this, we use the factorization formula as in (3.13)–(3.14). Define $\widetilde{Y}_N$ as

$$\widetilde{Y}_N(t,x) = \frac{1}{\sqrt{N}} \sum_{z\in\mathbb{Z}_N} \int_0^t (t-u)^{-\alpha} G_N(t-u,x,z) \mathbb{1}_{\{\tau_N\geq u\}} \sqrt{\widetilde{U}_N(u,z)} dB_z(u), \quad x \in \mathbb{Z}_N.$$

Note that

$$\widetilde{\eta}_N(t,x) = \frac{\sin(\pi\alpha)}{\pi} \frac{1}{N} \int_0^t \sum_{z\in\mathbb{Z}_N} (t-u)^{\alpha-1} G_N(t-u,x,z) \widetilde{Y}_N(u,z) du,$$

Throughout the remainder of this proof we take $q = 4 + 2\kappa$, with κ as Proposition 3.6, $1 < p = \frac{q}{q-1} < 2$, and $\alpha < \frac{1}{4}$, with additional restrictions on α to be added later. Proceeding as in (3.15) (see also [12] p. 791) we have by the Burkholder, Minkowski and Young inequalities

$$\begin{aligned} E\Big\{\|\widetilde{Y}_N(t,u,\cdot)\|_q^q\Big\} &\leq CE\Big\{\int_0^t (t-u)^{-2\alpha} \|G_N^2(t-u,\cdot,\cdot)\|_1 \|\mathbb{1}_{\{\tau_N\geq u\}} \widetilde{U}_N(u,\cdot)\|_{2+\kappa} du\Big\}^{\frac{q}{2}} \\ &\leq CR^{\frac{q}{2}} \Big\{\int_0^t (t-u)^{-2\alpha-\frac{1}{2}} du\Big\}^{\frac{q}{2}}. \end{aligned} \tag{4.4}$$

Applying now the Hölder and Minkowski inequalities, as well as (4.4) and (3.7) with ϱ and α chosen so that $0 < \frac{1}{2(4+2\kappa)} + \varrho < \alpha < \frac{1}{4}$, we have

$$
\begin{aligned}
E\Big\{ & \sup_{x,y\in\mathbb{Z}_N,\ s,t<T} |\widetilde{\eta}_N(t,x)-\widetilde{\eta}_N(s,y)|^q\Big\} \\
&\le C \sup_{x,y\in\mathbb{Z}_N,\ s,t<T} \Big[\int_0^t \|(t-u)^{\alpha-1}G_N(t-u,x,\cdot) \\
&\qquad -(s-u)^{\alpha-1}G_N(s-u,y,\cdot)\mathbb{1}_{(u\le s)}\|_p \Big[E\Big\{\|\widetilde{Y}_N(u,\cdot)\|_q^q\Big\}\Big]^{\frac{1}{q}} du\Big]^q \\
&\le C \sup_{x,y\in\mathbb{Z}_N,\ s,t<T} \Big(|t-s|^{\varrho}+|y-x|^{2\varrho}\Big)^q. \qquad (4.5)
\end{aligned}
$$

This bound and Markov's inequality show that we can choose δ so that (4.3) holds.

We now turn to the D_N term, for which $C1$ is trivial. To check $C2$ we shall show that, for $t\le \tau_N$, we can find δ such that $w_\delta^T(D_N)<\zeta$. This, combined with (4.2), will establish the required bound. Define the following subsets of $[0,T]^2\times\Omega$:

$$
A_1=\Big\{s<t\le\rho_N,\quad \tau\ge t\Big\},\ A_2=\Big\{s\le\rho_N\le t,\quad \tau\ge t\Big\},\ A_3=\Big\{\rho_N\le s<t,\quad \tau\ge t\Big\}.
$$

Let $x<y\in\mathbb{Z}_N$. On A_1, D_N does not depend on ρ_N, and so we have

$$
\begin{aligned}
\Big|D_N(t,x) & -D_N(s,y)\Big| \\
&=\Big|\int_0^t \frac{1}{N}\sum_{z\in\mathbb{Z}_N}\Big(G'_N(t-u,x,z)-G'_N(s-u,y,z)\mathbb{1}_{(u\le s)}\Big)F_N(U_N)(u,z)du\Big| \\
&\le \Big|\int_0^s \frac{1}{N}\sum_{z\in\mathbb{Z}_N} G'_N(t-u,x,z)-G'_N(s-u,y,z)F_N(U_N)(u,z)du\Big| \\
&\qquad +\int_s^t \frac{1}{N}\sum_{z\in\mathbb{Z}_N}\Big|G'_N(t-u,x,z)F_N(U_N)(u,z)\Big|du \\
&\triangleq I_1(s)+I_2(s,t).
\end{aligned}
$$

The variables s and t in the definition of I_1 and I_2 refer only on the range of integration, not on their appearance in the integrands. We first bound I_1.

$$
\begin{aligned}
I_1(s)=&\Big|\int_0^s \frac{1}{N}\sum_{z\in\mathbb{Z}_N}\Big(G_N(t-\tfrac{u+s}{2},x,z)-G_N(\tfrac{s-u}{2},y,z)\Big)X_N(s,u,z)du\Big| \\
&\le \int_0^s \|G_N(t-\tfrac{u+s}{2},x,\cdot)-G_N(\tfrac{s-u}{2},y,\cdot)\|_p\,\|X_N(s,u,\cdot)\|_q du,
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$
X_N(s,u,z)=\frac{1}{N}\sum_{\varsigma\in\mathbb{Z}_N} G'_N(\tfrac{s-u}{2},\varsigma,z)F_N(U_N)(u,\varsigma).
$$

Using Young's inequality with $1+\frac{1}{q}=\frac{1}{\beta}+\frac{1}{1+\kappa/2}$ and (3.5) we have

$$\|X_N(s,u,\cdot)\|_q \le C\|G_N'(\tfrac{s-u}{2},\cdot,0)\|_\beta \,\|U_N(u,\cdot)\|^2_{2+\kappa} \le CR^2(\frac{s-u}{2})^{-1+\frac{1}{2\beta}}.$$

Now choose $\varrho,\ 0<\varrho<\frac{1}{2}(\frac{1}{\beta}-\frac{1}{q})=\frac{1}{2}(1-\frac{1}{2+\kappa})$. Then, by (3.6), we have

$$\begin{aligned} I_1 \le & CR\Big(|y-x|^{2\varrho}+|t-s|^{\varrho}\Big)\int_0^s (s-u)^{-1+\frac{1}{2}(\frac{1}{\beta}-\frac{1}{q})-\varrho}du \\ \le & CR\Big(|y-x|^{2\varrho}+|t-s|^{\varrho}\Big). \end{aligned}$$

We now estimate I_2. With the same notation as for I_1, we have

$$\begin{aligned} I_2 = & \Big|\int_s^t \frac{1}{N}\sum_{z\in\mathbb{Z}_N} G_N(\tfrac{t-u}{2},x,z)X_N(t,u,z)du\Big| \\ \le & \int_s^t \|G_N(\tfrac{t-u}{2},x,\cdot)\|_p\|X_N(t,u,\cdot)\|_q du \\ \le & C\int_s^t (t-u)^{-\frac{1}{2}+\frac{1}{2p}-1+\frac{1}{2\beta}}du \\ \le & C|t-s|^{\varrho}. \end{aligned}$$

Therefore, on A_1, we have

$$\Big|D_N(t,x)-D_N(s,y)\Big| \le C\Big(|y-x|^{2\varrho}+|t-s|^{\varrho}\Big). \tag{4.6}$$

On A_2, with the same notation as above, $\Big|D_N(t,x)-D_N(s,y)\Big| \le I_1(s)+I_2(s,\rho_N)$, so that I_1 can be bounded as for the situation A_1 and, for I_2, we are led to

$$I_2(s,\rho_N) \le C\int_s^{\rho_N}(t-u)^{-\frac{1}{4}}(t-u)^{-1+\frac{1}{2q}}du \le C|t-s|^{\varrho}.$$

On A_3, $I_2=0$ and

$$\begin{aligned} \Big|D_N(t,x)-D_N(s,y)\Big| \le & I_1(\rho_N) \\ \le & \int_0^{\rho_N} \|G_N(t-\tfrac{u+s}{2},x,\cdot)-G_N(\tfrac{s-u}{2},y,\cdot)\|_2\,\|X_N(s,u,\cdot)\|_2du \\ \le & \int_0^{s} \|G_N(t-\tfrac{u+s}{2},x,\cdot)-G_N(\tfrac{s-u}{2},y,\cdot)\|_2\,\|X_N(s,u,\cdot)\|_2du \\ \le & C\Big(|y-x|^{2\varrho}+|t-s|^{\varrho}\Big). \end{aligned}$$

In conclusion, we have the bound (4.6) on $\bigcup A_i = \{\sup_{t\le T}\|\widetilde{U}_N\|_{2+\kappa} < R\}$ and for any $0\le s<t<T$, $x<y\in\mathbb{Z}_N$. Since this bound depends on t,s,x and y only through the increments, we can find δ such that (4.6) is uniformly bounded by ζ for $|y-x|\vee|t-s|\le\delta$ and $s,t\le T$, and we are done. $\square$

Corollary 4.3 *Let ρ_N be as in* (3.18). *Then $\rho_N \to \infty$ in probability. Moreover, the sequence of polygonal interpolations to the U_N of* (2.1) *is tight in C and all its limit points are non-negative.*

PROOF. Since $P\{\tilde{U}_N(t,x) = U_N(t,x),\ t \le \rho_N, x \in \mathbb{Z}_N\} = 1$, the second statement follows from the first one and Theorems 2.2 and 4.2. Our bounds (4.5) and (4.6) on the modulus of continuity of $\tilde{U}_N$ suffice to prove the first statement.

5 Existence of the Burgers superprocess

We now turn to the main task of this paper, that of establishing the existence of a solution to (1.1) – the Burgers superprocess. Due to the presence of the space-time white noise, (1.1) should be written in the weak form,

$$\begin{aligned}\int_{\mathbb{R}} u(t,x)\varphi(x)dx = &\int_{\mathbb{R}} u(0,x)\varphi(x)dx \\ &+ \int_0^t \int_{\mathbb{R}} u(s,x)\varphi''(x)dxds \ + \ \int_0^t \int_{\mathbb{R}} u^2(s,x)\varphi'(x)dsdx \\ &+ \int_0^t \int_{\mathbb{R}} \sqrt{u(s,x)}\varphi(x)\, W(ds,dx),\end{aligned} \tag{5.1}$$

for $\varphi \in C_c^2$, the space of C^2 functions on $\mathbb{R}$ with compact support. Assume that the initial value f (deterministic for simplicity) satisfies the conditions of Theorem 1.1. Then, applying now standard techniques (cf. [14,25] – details are given in [2]) it is not hard to show that the SPDE (5.1) is equivalent to the following martingale problem.

$$\begin{cases}\text{For any } \varphi \in C_c^2 \\ M_\varphi(t) := \langle u(t,\cdot),\varphi\rangle - \langle u(0,\cdot),\varphi\rangle - \int_0^t \langle u(s,\cdot),\varphi''\rangle - \langle u^2(s,\cdot),\varphi'\rangle ds \\ \text{ is an } \mathcal{F}_t \text{ square integrable martingale with quadratic variation} \\ \langle M_\varphi\rangle_t = \int_0^t \langle u(s,\cdot),\varphi^2\rangle\, ds.\end{cases} \tag{5.2}$$

We can now finally complete the proof of the central Theorem 1.1. What remains of the proof, for which we shall skip some details, is quite standard (see e.g. [19]) and is based on showing that the approximating processes U_N converge to the solution of the equivalent martingale problem (5.2).

PROOF OF THEOREM 1.1. By Corollary 4.3, we can take a weakly converging subsequence of polygonal interpolations of U_N, which we again denote by $\bar{U}_N$. By a standard Skorokhod embedding argument, we can find a probability space such that the convergence is with probability one. We shall assume that we are on that space and denote the limit by u. By Corollary 4.3, $u(t,x) \ge 0$. Moreover, by Fatou's Lemma and the bounds of Section 3, $u(t,\cdot) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ a.s. For $\varphi \in C_c^2$ we define

$$
\begin{aligned}
Z_N^{\varphi}(t) = &\frac{1}{N}\sum_{x\in\mathbb{Z}_N}\varphi(x)\bar{U}_N(t,x) - \frac{1}{N}\sum_{x\in\mathbb{Z}_N}\varphi(x)f_N(x)\\
&- \int_0^t \frac{1}{N}\sum_{x\in\mathbb{Z}_N}\varphi(x)\Delta_N\bar{U}_N(s,x)ds - \int_0^t \frac{1}{N}\sum_{x\in\mathbb{Z}_N}\varphi(x)\nabla_N F_N(\bar{U}_N)(s,x)ds\\
= &\int_0^t \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}_N}\varphi(x)\sqrt{\bar{U}_N(s,x)}dB_x(s). \qquad (5.3)
\end{aligned}
$$

Since Z_N^{φ} is a square integrable martingale with $\langle Z_N^{\varphi}\rangle_t = \int_0^t \frac{1}{N}\sum_{x\in\mathbb{Z}_N}\varphi(x)^2\bar{U}_N(s,x)\,ds$,

$$
Z_N^{\varphi}(t)^2 - \int_0^t \frac{1}{N}\sum_{x\in\mathbb{Z}_N}\varphi(x)^2\bar{U}_N(s,x)\,ds \qquad (5.4)
$$

is a martingale. Since the $\bar{U}_N$ converge almost surely, it follows from $\varphi \in C_c^2$ that each term on the right hand side of (5.3) converges as $N \to \infty$ and so $Z_N^{\varphi}(t)$ converges almost surely to a local martingale $Z^{\varphi}(t)$. Since $Z_N^{\varphi}(t)$ is bounded in L^2, the sequence $Z_N^{\varphi}(t)$ is uniformly integrable and so Z^{φ} is a (true) martingale (e.g. [13]). By expanding φ in a Taylor series, we find that $Z^{\varphi}(t)$ has the decomposition

$$
\begin{aligned}
Z^{\varphi}(t) = &\int \varphi(x)u(t,x)\,dx - \int \varphi(x)u(0,x)\,dx\\
&- \int_0^t\int u(s,x)\varphi''(x)\,dxds + \int_0^t\int u^2(s,x)\varphi'(x)\,dxds. \qquad (5.5)
\end{aligned}
$$

Furthermore, since the martingale (5.4) converges almost surely to a local martingale, the process $Z^{\varphi}(t)^2 - \int_0^t\int\varphi(x)^2\sqrt{u(s,x)}\,dxds$ is also a local martingale, which allows us to conclude that $\langle Z^{\varphi}\rangle_t = \int_0^t\int\varphi(x)^2\sqrt{u(s,x)}\,dxds$. ¿From this, we conclude that u solves the martingale problem (5.2) and so we have the required existence. The regularity property of the solution constructed by the approximation procedure follows from the bounds (4.5) and (4.6) in the proof of the tightness of $\bar{U}_N$. Finally, from (3.20) and Fatou's Lemma we get a similar bound for $u(t,\cdot)$. The fact that $\|u(t,\cdot\|_1$ is a Feller branching diffusion now follows from the argument used to proved the similar result for $\widetilde{U}_N(t,\cdot)$. It is well known that such diffusion processes die out in finite time.

6 On uniqueness

In the superprocess setting, the usual way to show uniqueness (in law) is to establish the existence of an appropriate dual process, as developed, for example, in [8]. We briefly sketch an attempt in this direction.

For $v \in L^1$, $\phi \in C$ and $\pi \in \mathbb{N}$, consider test functions of the form

$$
F_{\phi,\pi}(v) = \int_{\mathbb{R}^{\pi}}\phi(x_1,\ldots,x_{\pi})\prod_{i=1}^{\pi}v(x_i)dx_i, \qquad (6.1)
$$

and define for $\phi \in C_c^2$

$$\Theta_i^1(\phi)(x_1, \dots, x_{\pi+1}) = \frac{\partial \phi(x_1, \dots, x_\pi)}{\partial x_i} \delta(x_i - x_{\pi+1}),$$
$$\Theta_{i,j}^2(\phi)(x_1, \dots, x_{\pi-1}) = \phi(x_1, \dots, x_{j-1}, x_i, x_{j+1} \dots x_{\pi-1}) \quad j > i. \tag{6.2}$$

With these notations, and using the weak equation (5.1), one can check that

$$\begin{aligned} F_{\phi,\pi}(u_t) &= F_{\phi,\pi}(u_0) \\ &\quad + \int_0^t F_{\mathbf{\Delta}\phi,\pi}(u_s) ds + \int_0^t \int_{\mathbb{R}^\pi} \sum_{i=1}^{\pi} \Big(F_{\Theta_i^1(\phi),\pi+1}(u_s) - F_{\phi,\pi}(u_s) \Big) ds \\ &\quad + \int_0^t \int_{\mathbb{R}^\pi} \sum_{\substack{1 \le i,j \le \pi \\ j \ne i}} \Big(F_{\Theta_{ij}^2(\phi),\pi-1}(u_s) - F_{\phi,\pi}(u_s) \Big) ds \\ &\quad + \int_0^t \tfrac{1}{2}\pi(\pi+1) F_{\phi,\pi}(u_s) ds + \text{ a martingale.} \end{aligned} \tag{6.3}$$

This suggests defining a dual process (ϕ_t, π_t) as follows: The process $\{\pi_t,\ t \ge 0\}$ is a birth and death process with birth rate $\lambda \pi_t$ and death rate $\frac{1}{2}\pi_t(\pi_t + 1)$. When π_t jumps up, ϕ_t jumps to $\Theta_i^1(\phi_t),\ i \in \{1, \dots \pi_t\}$. When π_t jumps down, ϕ_t jumps to $\Theta_{i,j}^2(\phi_t),\ i,j \in \{1, \dots \pi_t\}, j > i$. Between the jumps, ϕ_t solves the heat equation in R^{π_t}. Then, the 'duality equation', or Feyman-Kac formula, is

$$E\Big\{ F_{\phi,\pi}(u_t) \Big\} = E\Big\{ F_{\phi_t,\pi_t}(u_0) e^{\int_0^t \frac{1}{2}\pi_s(\pi_s+1) ds} \Big\}. \tag{6.4}$$

To make all this work, one has to show that each of the expressions in (6.4) is finite when absolute values are taken inside the expectation. We have not been able to do so and the uniqueness question for the Burgers superprocess thus remains open.

7 Appendix

PROOF OF LEMMA 3.1. Recall ([9] p. 567) that the characteristic function of the continuous time simple random walk ξ_t on $\mathbb{Z}$ is

$$\phi(\omega, t) = E\Big\{ e^{i\omega\xi_t} \Big\} = e^{t(\cos(\omega)-1)}, \qquad \omega \in [-\pi, \pi]. \tag{7.1}$$

Using (7.1), it is easy to show that the Fourier transform $\widehat{G}_N$ of G_N is

$$\widehat{G}_N(t, x, \omega) = e^{i\omega x} e^{N^2 t(\cos(\omega/N)-1)}, \qquad \omega \in [-N\pi, N\pi],\ x \in \mathbb{Z}_N. \tag{7.2}$$

Proof of (3.4). Using the Fourier transform and the inequality $-0.5x^2 \le \cos x - 1 < -0.2x^2$ for $x \in [-\pi, \pi]$, we can show (3.4) for $p = 2$ using Plancherel's formula

$$\|G_N(t,x,.)\|_2^2 = \int_{-N\pi}^{N\pi} e^{2N^2t(\cos(\omega/N)-1)}d\omega \le \int_{-N\pi}^{N\pi} e^{-0.4t\omega^2}d\omega \le \frac{3}{\sqrt{t}}. \tag{7.3}$$

For $p = 1$, the bound is of course 1, and for $1 < p < 2$ it can be obtained by interpolation between L^p spaces in the following form: If $f \in L^p \cap L^q$, $1 \le p \le q \le \infty$,

$$\|f\|_r \le \|f\|_p^\theta \|f\|_q^{1-\theta} \quad \text{where} \quad p \le r \le q, \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, \quad \theta = \frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}. \tag{7.4}$$

Proof of (3.5). Note that by symmetry and since G_N is decreasing in $|x|$,

$$\|G_N'(t,x,.)\|_1 = 2G_N(t,0,0) = 2\|G_N(t/2,x,\cdot)\|_2^2 \le Ct^{-1/2}. \tag{7.5}$$

The bound for $p \ge 2$ follows as in (7.8) below, but uses the Hausdorff-Young inequality for $p > 2$, the case $1 < p < 2$ coming via interpolation.

Proof of (3.6). We first establish the bound given in (3.6) for $p = 2$. Note that

$$\begin{aligned} &\|G_N(t,x,\cdot) - G_N(t+\delta),z,\cdot)\|_2 \\ &\quad \le \|G_N(t,x,\cdot) - G_N(t,z,\cdot)\|_2 + \|G_N(t,z,\cdot) - G_N(t+\delta),z,\cdot)\|_2. \end{aligned} \tag{7.6}$$

We evaluate the time increments in (7.6) using Plancherel's formula, the mean value theorem and the Minkowski and Jensen inequalities as follows:

$$\begin{aligned} &\Big[\int_{-N\pi}^{N\pi}\Big(e^{N^2(t+\delta)(\cos(\lambda/N-1)} - e^{N^2t(\cos(\lambda/N)-1)}\Big)^2 d\lambda\Big]^{1/2} \\ &\le \Big\{\int_{-N\pi}^{N\pi}\Big(\int_t^{t+\delta} N^2|\cos(\lambda/N)-1|e^{N^2u(\cos(\lambda/N)-1)}du\Big)^2 d\lambda\Big\}^{1/2} \\ &\le \int_t^{t+\delta}\Big[\int_0^{N\pi}\lambda^4 e^{-0.4u\lambda^2}d\lambda\Big]^{1/2}du \\ &< \int_t^{t+\delta} u^{-5/4}\Big[\int_0^\infty \lambda^4 e^{-0.4\lambda^2}d\lambda\Big]^{1/2}du \\ &\le C\delta\Big[\int_t^{t+\delta} u^{-\frac{5}{4\alpha}}\frac{1}{\delta}du\Big]^\alpha \\ &\le C\delta^\varrho t^{-\frac14-\varrho}. \end{aligned} \tag{7.7}$$

For the spatial increment we use the inequality $|1 - e^{i\lambda(x-z)}|^2 \le 4|\lambda|^\epsilon|x-z|^\epsilon$ for $0 < \epsilon < 1$ to see that

$$\begin{aligned} \|G_N(t,x,\cdot) - G_N(t,y,\cdot)\|_2 &\le \Big[\int_{-N\pi}^{N\pi}\Big|e^{i\lambda x} - e^{i\lambda y}\Big|^2 e^{2N^2t(\cos(\lambda/N-1)}d\lambda\Big]^{1/2} \\ &\le C|y-x|^{2\varrho}t^{-\frac{4\varrho+1}{4}}\Big[\int_0^\infty \lambda^{4\varrho}e^{-0.4\lambda^2}d\lambda\Big]^{1/2} \\ &\le C|y-x|^{2\varrho}t^{-\frac14-\varrho}. \end{aligned} \tag{7.8}$$

This finishes the proof for $p=2$. For $p=1$, we have by the Cauchy problem for G_N

$$
\begin{aligned}
\|\frac{\partial}{\partial t}G_N(t,x,\cdot)\|_1 &= \|\Delta_N G_N(t,x,\cdot)\|_1 \\
&= \Big\|\frac{1}{N}\sum_{z\in\mathbb{Z}_N}\Big(G_N(t/2,x+\tfrac{1}{N},z)-G_N(t/2,x,z)\Big) \\
&\qquad\qquad \times\Big(G_N(t/2,z-1,\cdot)-G_N(t/2,z,\cdot)\Big)\Big\|_1 \\
&\le \|G'(t/2,x,\cdot)\|_1^2.
\end{aligned}
$$

This estimate, (7.5), the mean value theorem and the last lines of (7.7) complete the proof. For the spatial increments (and $p=1$), we have, by (7.5), that

$$
\|G_N(s,x,\cdot)-G_N(s,y,\cdot)\|_1 \le |y-x|\|G'(s,x,\cdot)\|_1 \le C|y-x|s^{-1/2}.
$$

However, we also have $\|G_N(s,x,\cdot)-G_N(s,y,\cdot)\|_1\le 2$, and so for $0<\varrho<1$,

$$
\|G_N(s,x,\cdot)-G_N(s,y,\cdot)\|_1 \le C|y-x|^{2\varrho}s^{-\varrho}.
$$

This establishes (3.6) for $p=1$. The case $1<p<2$ follows by interpolation.

Proof of (3.7). For p, α and ϱ as required, and $0<s<t$, we have, by (3.4) and (3.6),

$$
\begin{aligned}
&\int_0^t \|(t-u)^{\alpha-1}G_N(t-u,\cdot,x)-(s-u)^{\alpha-1}G_N(s-u,\cdot,y)\mathbb{1}_{(u\le s)}\|_p du \\
&\quad\le \int_0^s \Big((t-u)^{\alpha-1}-(s-u)^{\alpha-1}\Big)\|G_N(t-u,\cdot,x)\|_p du \\
&\qquad + \int_0^s (s-u)^{(\alpha-1)}\|G_N(t-u,\cdot,x)-G_N(s-u,\cdot,y)\|_p du \\
&\qquad + \int_s^t (t-u)^{(\alpha-1)}\|G_N(t,\cdot,x)\|_p du \\
&\quad\le C\int_0^s \Big((t-u)^{\alpha-1}-(s-u)^{\alpha-1}\Big)(t-u)^{-\frac{1}{2}(1-\frac{1}{p})}du \\
&\qquad + C\Big(|y-x|^{2\varrho}+|t-s|^{\varrho}\Big)\int_0^s (s-u)^{\alpha-1-\frac{1}{2}-\varrho+\frac{1}{2p}}du \\
&\qquad + C(t-s)^{\alpha-\frac{1}{2}(1-\frac{1}{p})}.
\end{aligned}
$$

Note that for $\sigma<0$, $0<\varrho<1$, there is a positive constant C such that $t^\sigma-s^\sigma\le C(t-s)^\varrho s^{\sigma-\varrho}$. Using this and $(t-u)^\sigma<(s-u)^\sigma$ for $\sigma<0$ we bound the first term in the last inequality above, the other two are trivial. □

Lemma 7.1 *If $f,g\in l_N^2$ then*

$$
\frac{1}{N}\sum_{x\in\mathbb{Z}_N} g(x)f(x)\nabla_N f(x) \le \|\nabla_N f\|_2^2+\frac{1}{8}\|f\|_2^2\|g\|_2^4. \tag{7.9}
$$

PROOF. In the sequel we will need a discrete version of Sobolev inequality: For any $f \in l_N^2$, it holds that

$$\|f\|_\infty^2 \leq \|\nabla_N f\|_2 \|f\|_2. \tag{7.10}$$

To show this inequality note that, for any $N \in \mathbb{N}$ and $f \in l_N^2$,

$$f^2(x) = \frac{1}{2N}\left[\sum_{\substack{z\in\mathbb{Z}_N\\ z=-\infty}}^{x-1} \nabla_N^+ f^2(x) - \sum_{\substack{z\in\mathbb{Z}_N\\ z=x+1}}^{\infty} \nabla_N^- f^2(x)\right].$$

However,

$$\begin{aligned}\nabla_N^+ f^2(x) &= N\Big(f^2(x+\tfrac{1}{N}) - f^2(x)\Big)\\ &= Nf(x)\Big[f(x+\tfrac{1}{N}) - f(x)\Big] + Nf(x+\tfrac{1}{N})\Big[f(x+\tfrac{1}{N}) - f(x)\Big],\end{aligned}$$

with a similar expression for ∇_N^-. The conclusion then follows from the Cauchy–Schwartz inequality.

To complete the proof of Lemma 7.1, we can assume $\|f\|_2^2 > 0$. By Hölder's inequality

$$\begin{aligned}\frac{1}{N}\sum_{x\in\mathbb{Z}_N} g(x)f(x)\nabla_N f(x) &\leq \|f\|_\infty \|\nabla_N f\|_2 \|g\|_2\\ &\leq \frac{1}{2}\|\nabla_N f\|_2\left[\frac{\|f\|_\infty^2}{\|f\|_2} + \|f\|_2\|g\|_2^2\right]\\ &\leq \frac{1}{2}\|\nabla_N f\|_2^2 + \|\nabla_N f\|_2 \frac{\|f\|_2\|g\|_2^2}{2}\\ &\leq \|\nabla_N f\|_2^2 + \frac{1}{8}\|f\|_2^2\|g\|_2^4,\end{aligned}$$

by (7.10) and exploiting the elementary inequality $ab \leq 1/2(\epsilon a^2 + 1/\epsilon b^2)$ for $\epsilon > 0$.$\square$

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