Part III

The Geometry of Random Fields

You could not possibly have got this far in our book without having read the preface, so you already know that you have finally reached the main part of this book. It is here that the important results lie and it is here that, after over 250 pages of preparation, there will also be new results.

With the general theory of Gaussian processes behind us in Part I, and the geometry of Part II now established, we return to the stochastic setting.

There are three main (classes of) results in this part. The first is an explicit formula for the expected Euler characteristic of the excursion sets of smooth Gaussian random fields. In the same way that we divided the treatment of the geometry into two parts, Chapter 11 will cover the theory of random fields defined over simple Euclidean domains and Chapter 12 will cover fields defined over Whitney stratified manifolds. Unlike the case in Chapter 6, however, even if you are primarily interested in the manifold scenario you will need to read the Euclidean case first, since some of the manifold computations will be lifted from this case via atlas-based arguments.

As an aside, in the final section (Section 12.6) of Chapter 12 we shall return to a purely deterministic setting and use our Gaussian field results to provide a probabilistic proof of the classical Chern–Gauss–Bonnet theorem of differential geometry using nothing[22] but Gaussian processes. This really has nothing to do with anything else in the book, but we like it too much not to include it.

In Chapter 13 we shall see how to lift the results about the mean Euler characteristics of excursion sets to results about mean Lipschitz–Killing curvatures. The argument will rely on a novel extension of the classical Crofton formulas about averaged cross-sections of Euclidean sets to a scenario in which the cross-sections are replaced by intersections of the set with certain random manifolds, and the averaging is against a Gaussian measure. This new "Gaussian Crofton" formula is completely new and would seem to be of significant independent interest.

The second main result appears in Chapter 14, where we shall finally prove the result promised long ago, that not only is the difference

$$\left| \mathbb{P}\left\{ \sup_{t\in M} f(t) \geq u \right\} - \mathbb{E}\{\varphi(A_u(f,M))\} \right|$$

extremely small for large u, but it can even be bounded in a rigorous fashion. Not only will this justify our claims that the mean Euler characteristic of excursion sets yields an excellent approximation to excursion probabilities, but, en passant, we shall also show that the volume-of-tubes approximation of Chapter 10 can be made rigorous as well.

In the closing Chapter 15 we finally leave the Gaussian scenario and develop the third main result, an explicit formula for both the expected Euler characteristics

[22] Of course, this cannot really be true. Establishing the Chern–Gauss–Bonnet theorem without any recourse to algebraic topology would have been a mathematical coup that might even have made probability theory a respectable topic within pure mathematics. What will be hidden in the small print is that everything relies on the Morse theory of Section 9.3, and this, in turn, uses algebraic geometry. However, our approach will save myopic probabilists from having to read the small print.

and Lipschitz–Killing curvatures of excursion sets for a wide range of non-Gaussian random fields, working in the general environment of differential, rather than integral, geometry. The proof and approach of this chapter are quite different from those of Chapters 11–13, and much of it is entirely new.

In fact, the results of Chapter 15 will actually incorporate most of the results of Chapters 11–13, using only a small part of what is developed there. Consequently, you could actually skip these chapters and go straight to the punch line in Chapter 15, although you will, occasionally, have to backtrack to pick up some material from the missed chapters. In doing so, however, you will miss a lot of interesting and useful additional material, and so we advise against this.